1, 2, 3

THE LAST 3 PAGES OF THE 2/13 NOTES WERE COVERED TODAY (2/15).

SPECIFICALLY:

THM: $f : X \to \mathbb{C}^*$ CONTINUOUS WITH X SIMPLY CONNECTED & PATH CONNECTED $\Rightarrow$ THERE IS A CONTINUOUS LIFT UNDER exp $\tilde{f} : X \to \mathbb{C}$ WITH $f = \exp(\tilde{f})$ WHICH IS UNIQUE UP TO ADDITION OF $2\pi i k$, $k \in \mathbb{Z}$

(AND PROOF)

COR: $f \in \mathcal{O}(U)$ WITH $f(z) \neq 0$ in U (U SIMPLY CONNECTED DOMAIN)

- $\exists\, g \in \mathcal{O}(U)$ SO THAT $f = \exp(g)$ [UNIQUE UP TO $2\pi i k$, $k \in \mathbb{Z}$]
- $\exists\, h \in \mathcal{O}(U)$ FOR EACH $n \in \mathbb{Z}^+$ SO THAT $f = h^n$ [UNIQUE UP TO MULTIPLE OF nTH ROOT OF UNITY]

WINDING NUMBERS

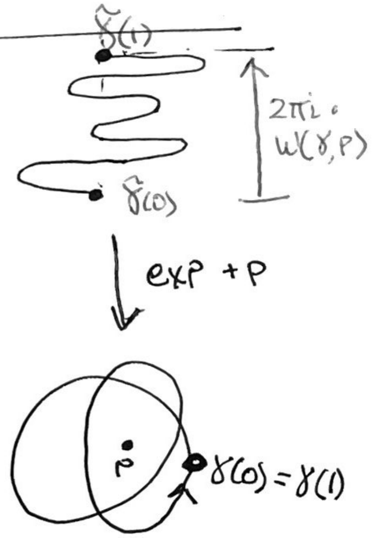
USING THE IDEA OF LIFTING OF CURVES BY THE INVERSE OF EXP, WE CAN MAKE PRECISE HOW ~~....~~ MUCH A CURVE (CLOSED OR NOT) WRAPS AROUND A POINT $p \in \mathbb{C}$.

DEF: LET $\gamma : [0,1] \to \mathbb{C}$ BE A CURVE, AND p BE ANY POINT IN $\mathbb{C} \setminus \{\gamma\}$. LET $\tilde{\gamma}$ BE ANY LIFT OF γ UNDER $z \mapsto \exp(z) + p$.

$$W(\gamma, p) = \frac{1}{2\pi i}\left(\tilde{\gamma}(1) - \tilde{\gamma}(0)\right)$$

IS THE <u>WINDING NUMBER OF γ WITH RESP. TO p</u>.

$W(\gamma, p) \in \mathbb{Z} \iff \gamma$ IS A CLOSED CURVE



PROPERTIES OF $W(\gamma, p)$

i) INVARIANT UNDER TRANSLATION:

FOR ANY $w \in \mathbb{C}$, $W(\gamma, p) = W(\gamma + w, p + w)$

ii) INVERSE: $W(\gamma^{-}, p) = -W(\gamma, p)$

iii) ADDITIVE UNDER CONCATENATION: IF $\eta(0) = \gamma(1)$,

$$W(\gamma \cdot \eta, p) = W(\gamma, p) + W(\eta, p)$$

iv) HOMOTOPY INVARIANCE:

IF $\gamma \simeq \eta$ IN $\mathbb{C} \setminus \{p\}$, THEN $W(\gamma, p) = W(\eta, p)$

v) LET γ BE CLOSED.

γ AND η ARE FREELY HOMOTOPIC IN $\mathbb{C} \setminus \{p\}$ $\iff W(\gamma, p) = W(\eta, p)$

vi) LET γ BE CLOSED.

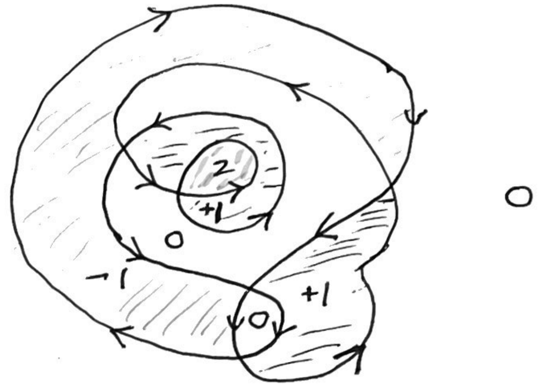
$z \mapsto W(\gamma, z)$ IS CONSTANT ON EACH COMPONENT OF $\mathbb{C} \setminus \{\gamma\}$ AND VANISHES ON THE UNBOUNDED COMP.

(PROOFS ARE STRAIGHTFORWARD & SO OMITTED.)

THIS LAST GIVES RISE TO

THE JUMP PRINCIPLE:

EACH TIME A CLOSED CURVE γ IS CROSSED FROM RIGHT TO LEFT, $W(\gamma, z)$ INCREASES BY $+1$



A MORE PRECISE STATEMENT TAKES A LITTLE WORK.

DEF A CLOSED CURVE $\gamma : [0,1] \to \mathbb{C}$ IS A <u>JORDAN CURVE</u> IF IT IS INJECTIVE ON $[0,1)$, IE γ INDUCES A HOMEOMORPHISM OF THE CIRCLE INTO $\mathbb{C}$

THE JORDAN CURVE THEOREM SAYS $\mathbb{C} \setminus \{\gamma\}$ HAS EXACTLY TWO COMPONENTS: A BOUNDED COMPONENT (THE <u>INTERIOR</u> OF γ) AND AN UNBOUNDED ONE (THE <u>EXTERIOR</u> OF γ), DENOTED $\mathrm{int}(\gamma)$ AND $\mathrm{ext}(\gamma)$ RESPECTIVELY.

FURTHER, $\partial(\mathrm{int}(\gamma)) = \{\gamma\} = \partial(\mathrm{ext}(\gamma))$.

CONSEQUENTLY IF WE HAVE A DISK $D \subset \mathbb{C}$ AND A CLOSED CURVE $\gamma : [0,1] \to \mathbb{C}$ SO THAT, FOR SOME $[a,b] \in D$,

- $\gamma(t) \in D$ FOR $a < t < b$
- $\gamma(t) \in \partial D \iff t = a$ OR $t = b$
- γ INJECTIVE ON $[a,b]$

THEN $\gamma : [a,b] \to D$ "DIVIDES" D INTO TWO PARTS, A RIGHT SIDE R AND A LEFT SIDE L.

(PROOF: ~~CON~~ CONSIDER $g : \overline{D} \to S^2$ A HOMEO. WHICH SENDS ∂D TO A POINT,

THEN $(g\circ\gamma)[a,b]$ IS A JORDAN CURVE, SEPARATING THE SPHERE INTO TWO COMPONENTS, CORRESPONDING TO THE "LEFT" SIDE" AND THE "RIGHT SIDE".

~~THE R~~ IF ∂D IS ORIENTED COUNTERCLOCKWISE THEN THE BOUNDARY OF THE RIGHT ~~GOES~~ CONSISTS OF ∂D FROM $\alpha = \gamma(a)$ TO $\beta = \gamma(b)$ FOLLOWED BY γ^{-} FROM β TO α. L IS DEFINED SIMILARLY.

NOTE IF $p \in L$ AND $q \in R$, THE JUMP PRINCIPLE SAYS

$$\boxed{W(\gamma, p) = 1 + W(\gamma, q) \quad \text{FOR } p \in L,\ q \in R}$$

THM: IF $\gamma : [0,1] \to \mathbb{C}$ IS A JORDAN CURVE

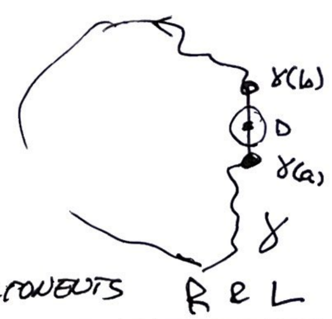
$$W(\gamma, p) = \begin{cases} \pm 1 & \text{IF } p \in \operatorname{int}(\gamma) \\ 0 & \text{IF } p \in \operatorname{ext}(\gamma) \end{cases}$$

γ IS POSITIVELY ORIENTED $\iff W(\gamma, p) = +1$ FOR ANY $p \in \operatorname{int}\gamma$

Pf/ SINCE $W(\gamma, p) = 0$ ~~OUTSIDE~~ FOR $p \in \operatorname{ext}(\gamma)$, WE NEED ONLY CONSIDER $p \in \operatorname{int}(\gamma)$.

NOW FIRST REDUCE TO THE CASE WHERE $\{\gamma\}$ CONTAINS A STRAIGHT SEGMENT $[\gamma(a), \gamma(b)] \subset \mathbb{C}$.

LET D BE A DISK CENTERED AT THE MIDPOINT OF THIS SEGMENT.



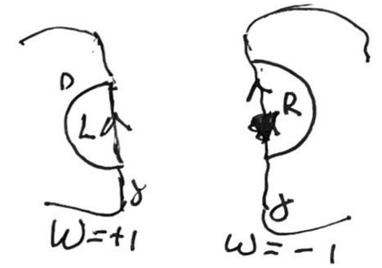
IF D IS SUFF. SMALL, THE JUMP PRINCIPLE APPLIES SINCE $D \setminus \{\gamma\}$ HAS TWO COMPONENTS R & L

EXACTLY ONE OF R OR L MUST LIE IN $ext(\gamma)$

ON THAT COMPONENT $W(\gamma,\cdot) = 0$.

IF $L \in int(\gamma)$, $W(\gamma,\cdot) = +1$ ON L

IF $R \in int(\gamma)$ $W(\gamma,\cdot) = -1$ ON R



FOR THE GENERAL CASE, WE CAN DEFINE A HOMOTOPY FROM γ TO CURVES OF THE FORM IN THE PREVIOUS ARGUMENT.... I'LL SKIP THE DETAILS HERE.

IF γ IS PIECEWISE $\mathbb{C}^1$, THE WINDING NUMBER CAN BE DEFINED ANALYTICALLY:

THM LET $\gamma : [0,1] \to \mathbb{C}$ BE PIECEWISE $\mathbb{C}^1$ WITH $p \in \mathbb{C} \setminus \{\gamma\}$. THEN

$$W(\gamma, p) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-p}$$

PF: SINCE γ IS $\mathbb{C}^1$, IT CAN BE LIFTED UNDER $\exp(z)+p$ TO GET A ~~PIECEWISE~~ $\tilde{\gamma}$ WHICH IS PIECEWISE $\mathbb{C}^1$, SINCE NEAR EACH $t \in [0,1]$, $\tilde{\gamma}$ IS THE COMPOSITION OF γ AND A (HOLOMORPHIC) LOCAL INVERSE OF $\exp(z)+p$.

THUS $\gamma'(t) = \exp(\tilde{\gamma}(t))\,\tilde{\gamma}'(t) = (\gamma(t)-p)\,\tilde{\gamma}'(t)$

EXCEPT AT FINITELY MANY t.

THUS $$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-p} = \frac{1}{2\pi i}\int_0^1 \frac{\gamma'(t)}{\gamma(t)-p}\,dt = \frac{1}{2\pi i}\int_0^1 \tilde{\gamma}'(t)\,dt$$

$$= \frac{1}{2\pi i}\left(\tilde{\gamma}(1)-\tilde{\gamma}(0)\right) = W(\gamma,p)$$