RECALL THAT FOR $z = x + iy$ WE HAVE

$$\exp(z) = e^z = e^x e^{iy} = e^x(\cos y + i \sin y).$$

HENCE $\exp(z) = \exp(\zeta) \iff \zeta = z + 2\pi i k$ FOR $k \in \mathbb{Z}$.

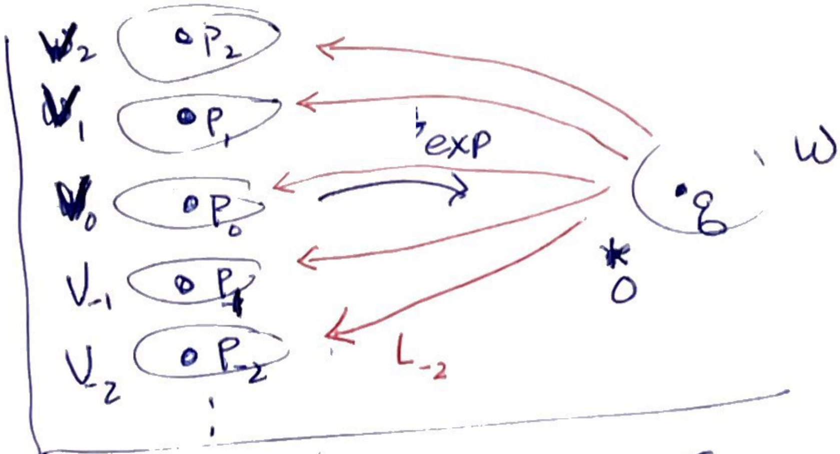
GIVEN ANY $q \in \mathbb{C} \setminus \{0\} = \mathbb{C}^*$, THERE IS A $p_0 \in \mathbb{C}$ WITH $\exp(p_0) = q$.

ACTUALLY, LOTS!

$$\exp^{-1}(q) = \{p \in \mathbb{C} \mid p = p_0 + 2\pi i k,\ k \in \mathbb{Z}\}.$$

SINCE $f(z) = e^z$ IS ENTIRE WITH $f'(z) \neq 0$ FOR $z \in \mathbb{C}$,

A NBHD W OF q WITH $0 \notin W$ IS EVENLY COVERED



BY NEIGHBORHOODS V_k OF p_k, WHERE

EACH $V_k = V_0 + 2\pi i k$ AND $\exp(p_0) = q$

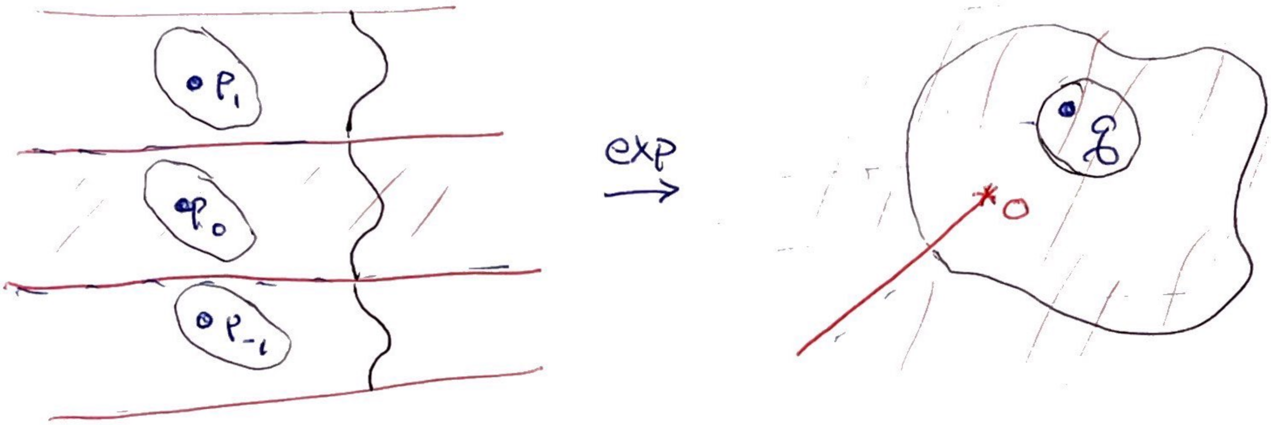
$p_k \in V_k$

WITH $\exp : V_k \to W$ A HOMEOMORPHISM (IN FACT A BIHOLOMORPHISM.)

THAT IS, WE CAN CONSTRUCT A LOCAL INVERSE MAP $L_k : W \to V_k$ FOR EACH $k \in \mathbb{Z}$

SO THAT $L_k(\exp(z)) = z$ ON V_k

IN FACT, WE CAN EXTEND W TO A SLIT PLANE



$W = \mathbb{C} - \{ tq \mid t \leq 0 \}$, AND IN THIS CASE

V_0 IS THE HORIZONTAL STRIP $\{ z \in \mathbb{C} \mid \mathrm{Im}(p_0) - \pi < z < \mathrm{Im}(p_0) + \pi \}$

AND THE OTHER V_k ARE VERTICAL TRANSLATES OF V_0.

SINCE W IS CONNECTED, EACH V_k IS A CONNECTED COMPONENT OF $\exp^{-1}(W)$.

EACH MAP $L_k : V_k \to W$ IS A BRANCH OF THE INVERSE (OF exp)

IN THIS CASE $L_k(w) = \log|w| + i\theta_k$

WHERE $\theta_k = \mathrm{Arg}(q)$

WITH

$$\mathrm{Im}(p_k) - \pi < \theta_k < \mathrm{Im}(p_k) + \pi$$

EACH L_k IS A BRANCH OF THE LOGARITHM.

MORE GENERALLY,

$$\begin{tikzcd} & \mathbb{C} \arrow[d, "\exp"] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & \mathbb{C}^* \end{tikzcd}$$

DEF

LET X BE A TOPOLOGICAL SPACE, WITH $f : X \to \mathbb{C}^*$ CONTINUOUS. THEN A CONTINUOUS MAP $\tilde{f} : X \to \mathbb{C}$ WHICH SATISFIES $\exp(\tilde{f}) = f$ IS A <u>LIFT</u> OF f (UNDER $\exp$)

<u>THM:</u> (UNIQUENESS) IF X IS A CONNECTED TOP. SPACE, ANY TWO LIFTS UNDER $\exp$ DIFFER BY AN INTEGER MULTIPLE OF $2\pi i$

Pf/ SPOSE $\tilde{f}_1, \tilde{f}_2 : X \to \mathbb{C}$ ARE TWO SUCH LIFTS OF $f : X \to \mathbb{C}^*$

FOR $p \in X$, $\exp(\tilde{f}_1(p)) = f(p) = \exp(\tilde{f}_2(p))$

SO THERE IS A $k_p \in \mathbb{Z}$ SO THAT $\tilde{f}_1(p) = \tilde{f}_2(p) + 2\pi i k_p$

THE FUNCTION $$p \to k_p = \frac{\tilde{f}_1(p) - \tilde{f}_2(p)}{2\pi i}$$

IS CONTINUOUS (SINCE THE f_j ARE) AND INTEGER VALUED. HENCE IT IS CONSTANT.

THM:

(i) LET $\gamma : [0,1] \to \mathbb{C}^*$ BE A CURVE WITH $\gamma(0) = \exp(p)$ FOR SOME $p \in \mathbb{C}$. THEN THERE IS A UNIQUE LIFT $\tilde{\gamma}$ OF γ SO THAT $\tilde{\gamma}(0) = p$. (CURVE LIFTING PROPERTY)

(ii) LET $H : [0,1] \times [0,1] \to \mathbb{C}^*$ BE CONTINUOUS, WITH $H(0,0) = \exp(p)$ FOR SOME $p \in \mathbb{C}$. THEN THERE IS A UNIQUE LIFT $\tilde{H} : [0,1] \times [0,1] \to \mathbb{C}$ SO THAT $\tilde{H}(0,0) = p$.

TO PROVE THIS, WE WILL USE

LEBESGUE'S COVER LEMMA: FOR EVERY OPEN COVER $\{U_\alpha\}$ OF A COMPACT METRIC SPACE X, THERE IS A $\delta > 0$, THEN IF $E \subset X$ SATISFIES $\operatorname{diam} E < \delta$, THERE IS A U_α SO THAT $E \subset U_\alpha$

(δ IS CALLED A LEBESGUE NUMBER FOR $\{U_\alpha\}$)

Pf/ EASY: FOR EACH $x \in X$, $\exists\ r > 0$ AND INDEX $\alpha(x)$ SO THAT $B_{2r}(x) = \{y \in X \mid |x-y| < 2r\}$ IS CONTAINED IN U_α.

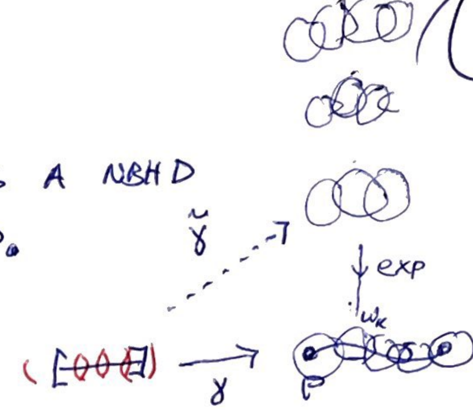
SINCE X IS COMPACT, $X = \bigcup_{i=0}^{n} B(x_i, r(x_i))$. LET $\delta = \min_{i=1..n} r(x_i)$

SINCE $E \subset X$ HAS $\operatorname{diam}(E) < \delta$ AND $B(x_j, r_j) \cap E \neq \emptyset$, FOR SOME x_j

$$E \subset B(x_j, 2r(x_j)) \subset U_{\alpha(x_j)}.$$ ∎

Pf of Thm:

(i) For each $t \in [0,1]$, $\gamma(t)$ has a nbhd that is evenly covered by exp.



By continuity, preimages under γ give an open cover of $[0,1]$. Let δ be a Lebesgue number for this cover and partition $[0,1]$ into $0 = t_0 < t_1 < t_2 < \cdots < t_n = 1$ with $|t_k - t_{k-1}| < 1$

Now $\gamma[t_{k-1}, t_k]$ is contained in an evenly covered neighborhood W_k for each k. $1 < k \leq n$.

Let O_1 be the component of $\exp^{-1}(W_1)$ containg $p = \gamma(0) = \gamma(t_0)$ and $L_1 : W_1 \to O_1$ be the local inverse.

Then define $\tilde{\gamma} = L_1 \circ \gamma$ on $[t_0, t_1]$.

Now let O_2 be the component of $\exp^{-1}(W_2)$ containing $\tilde{\gamma}(t_1)$
L_2 the corresponding local inverse, from $W_2 \to O_2$
and $\tilde{\gamma} = L_2 \circ \gamma$ on $[t_1, t_2]$

Continue in this way to construct $\tilde{\gamma} : [0,1] \to \mathbb{C}$.

This is unique by the uniqueness of lifts.

(ii) The proof here works almost the same: $H(t,s)$ has at every point $(t,s) \in [0,1] \times [0,1]$, a neighborhood evenly covered by exp.

We partition $[0,1] \times [0,1]$ into rectangles R_{jk} so that each has diam less than the Lebesgue # δ, to get neighborhoods W_{jk} that patch together...

The uniqueness of curves in (i) applied to curves made from the boundaries of R_{jk} ensure agreement on the edges & hence make it patch together...

COR SPOZE $\gamma_0 \simeq \gamma_1$ IN $\mathbb{C}^*$. TAKE $p \in \mathbb{C}$ SO THAT $\exp(p) = \gamma_0(0) = \gamma_1(0)$, AND LIFT TO $\tilde{\gamma}_0, \tilde{\gamma}_1$ WITH $\tilde{\gamma}_0(0) = \tilde{\gamma}_1(0) = p$.

THEN $\tilde{\gamma}_0 \cong \tilde{\gamma}_1$ AND $\tilde{\gamma}_0(1) = \tilde{\gamma}_1(1)$.

Pf/ LET H BE A HOMOTOPY BETWEEN γ_0 AND γ_1. WE GET A LIFT $\tilde{H}$ OF H WITH $H(0,0) = p$. AND LIFTS $\tilde{\gamma}_0, \tilde{\gamma}_1$ WITH $\tilde{\gamma}_0(0) = \tilde{\gamma}_1(0)$.

BY UNIQUENESS OF LIFTS
$$\gamma_0(t) = \tilde{H}(t,0)$$
$$\tilde{\gamma}_1(t) = \tilde{H}(t,1)$$

TO SEE $\tilde{H}$ IS A HOMOTOPY, OBSERVE THAT
$$\exp(\tilde{H}(1,s)) = H(1,s) = \gamma_0(1).$$

THE MAP $s \mapsto \tilde{H}(1,s)$ IS CONTINUOUS, AND TAKES VALUES IN THE SET $\exp^{-1}(\gamma_0(1))$
$$= \tilde{\gamma}_0(1) + 2\pi i \mathbb{Z}$$
WHICH IS DISCRETE.

COR A CLOSED CURVE IN $\mathbb{C}^*$ IS NULL-HOMOTOPIC $\iff$ EVERY LIFT OF IT UNDER exp IS A CLOSED CURVE.

Pf/ SPOZE γ_0 IS NULL-HOMOTOPIC IN $\mathbb{C}^*$, AND $\tilde{\gamma}_0$ BE ANY LIFT OF IT. NOW LET γ_1 BE THE CONSTANT CURVE $\gamma_1(t) = \gamma_0(0)$. THE LIFT $\tilde{\gamma}_1$ IS THE CONSTANT CURVE $\tilde{\gamma}_1(t) = \tilde{\gamma}_0(0)$.

SINCE $\gamma_0 \simeq \gamma_1$, $\tilde{\gamma}_0(1) = \tilde{\gamma}_1(1) = \tilde{\gamma}_0(0)$, SO $\tilde{\gamma}_0$ IS CLOSED.

$\Leftarrow$ ~~Spoze~~ SPOZE SOME LIFT $\tilde{\gamma}_0$ OF γ_0 IS A CLOSED CURVE.
(AND SO ALL)

SINCE THE PLANE IS SIMPLY CONNECTED $\tilde{\gamma}_0(t) \simeq \tilde{\gamma}_0(0)$ BY SOME HOMOTOPY $\tilde{H}$. $\exp \circ \tilde{H}$ GIVES A HOMOTOPY BETWEEN $\gamma_0(t)$ AND THE CONSTANT CURVE $\gamma_0(0)$.

$$\boxed{\underline{\text{COR:}} \quad \pi_1(\mathbb{C}^*, 1) \cong \mathbb{Z}}$$

Pf/ TAKE A HOMOTOPY CLASS $[\gamma] \in \pi_1(\mathbb{C}^*, 1)$ AND LIFT γ UNDER $\exp$ TO $\tilde{\gamma}$ WITH $\tilde{\gamma}(0) = 0 \in \exp^{-1}(1)$. THE END POINT $\tilde{\gamma}(1)$ IS OF THE FORM $0 + 2\pi i k$ $(k \in \mathbb{Z})$.

BY THE COROLLARY, k DOES NOT DEPEND ON THE CHOICE OF REPRESENTATIVE OF $[\gamma]$. THIS MEANS WE'VE CONSTRUCTED A MAP Φ WITH $\Phi([\gamma]) = k$ SO THAT $\Phi : \pi_1(\mathbb{C}^*, 1) \to \mathbb{Z}$.

THIS IS SURJECTIVE, SINCE $\Phi([e^{2\pi i k t}]) = k$ FOR ANY $k \in \mathbb{Z}$.

IT IS INJECTIVE, SINCE IF $\Phi([\gamma]) = 0$, $\tilde{\gamma}$ IS CLOSED AND HENCE NULL-HOMOTOPIC.

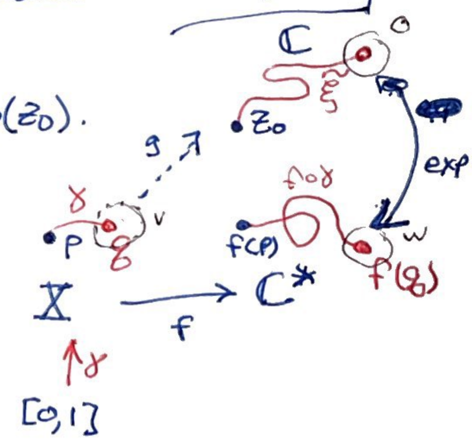
THUS Φ IS AN ISOMORPHISM.

THM: SPOZE X IS SIMPLY CONNECTED AND LOCALLY PATH CONNECTED, WITH $f: X \to \mathbb{C}^*$ CONTINUOUS. THEN THERE IS A CONTINUOUS $g: X \to \mathbb{C}$ SO THAT $f = \exp(g)$. THE MAP g IS UNIQUE UP TO ADDITION OF $2\pi i k$ $(k \in \mathbb{Z})$

$$\begin{tikzcd} & \mathbb{C} \arrow[d, "\exp"] \\ X \arrow[ur, dashed, "g"] \arrow[r, "f"'] & \mathbb{C}^* \end{tikzcd}$$

(NOTE THAT ANY OPEN CONNECTED SUBSET OF $\mathbb{C}$ IS PATH CONN.)

Pf/ SAY LAST: UNIQUENESS FOLLOWS DIRECTLY FROM EARLIER RESULTS, SO WE JUST NEED TO CONSTRUCT A LIFT.



- CHOOSE $p \in X$ AND $z_0 \in \mathbb{C}$ SO THAT $f(p) = \exp(z_0)$.

FOR ANY $\xi \in X$, LET γ BE A CURVE IN X WITH $\gamma(0) = p$, $\gamma(1) = \xi$.

THEN $f \circ \gamma$ LIFTS TO $\tilde{\xi} \in \mathbb{C}$ WITH $\tilde{\xi}(0) = z_0$

LET $g(\xi) = \tilde{\xi}(1)$. THIS IS WELL-DEFINED SINCE ANY CURVE IN $[f \circ \gamma]$ HAS THE SAME ENDPOINT.

THUS $\exp(g(\xi)) = \exp(\tilde{\xi}(1)) = f(\gamma(1)) = f(\xi)$.

- WE STILL HAVE TO CHECK THAT g IS CONTINUOUS, BUT THIS IS STRAIGHTFORWARD.

TAKE NBHD O OF $g(\xi)$ SMALL ENOUGH THAT $\exp$ IS A HOMEOMORPHISM ONTO A NEIGHBORHOOD $W = \exp(O)$ OF $f(\xi)$

BY CONTINUITY OF f & LOCAL PATH CONNECTEDNESS, THERE IS A NBHD V OF ξ SO THAT $f(V) \subset W$. FOR ANY $x \in V$, WE CAN FIND A PATH η FROM ξ TO x SO THAT $\eta(t) \in V$ AND HENCE $f \circ \eta$ IS A CURVE IN W FROM $f(\xi)$ TO $f(x)$. BUT W LIFTS UNIQUELY TO O, SO WE HAVE CONTINUITY.

COR: EXISTENCE OF LOGARITHMS & n^{th} ROOTS.

SPOZE $U \subset \mathbb{C}$ IS A SIMPLY CONNECTED DOMAIN AND $f \in \mathcal{O}(U)$ WITH $f(z) \neq 0$ IN U.

(i) THERE EXISTS $g \in \mathcal{O}(U)$ SO THAT $f = \exp(g)$. g IS UNIQUE UP TO ADDITION OF A MULTIPLE OF $2\pi i$

(ii) THERE EXISTS $h \in \mathcal{O}(U)$ FOR EACH $n \in \mathbb{Z}^+$ SUCH THAT $f = h^n$ AND h IS UNIQUE UP TO AN n^{th} ROOT OF UNITY

Pf/ (i) WE JUST NEED TO SHOW $g \in \mathcal{O}(U)$.

THIS IS STRAIGHTFORWARD: ~~RESTRICT~~ NEAR ANY $p \in U$ THERE IS AN EVENLY COVERED NEIGHBORHOOD W OF $f(p)$. LET W BE THE CONNECTED COMPONENT OF $\exp^{-1}(W)$ CONTAINING $g(p)$.

LET V BE THE CONN. COMPONENT OF $f^{-1}(W)$ CONTAINING p.

THE LOCAL INVERSE $L : W \to \mathcal{O}$ OF exp IS HOLOMORPHIC (SINCE exp IS) AND $L \circ f$ IS A LIFT OF f SENDING $p \mapsto g(p)$.

SINCE $g = L \circ f$ IN V, $g \in \mathcal{O}(V)$.

(ii) SIMILARLY, WRITE $f = \exp(g)$ FOR $g \in \mathcal{O}(U)$, AND LET $h = \exp(g/n)$. ~~THIS IS~~ $h \in \mathcal{O}(U)$ AND $f = h^n$.

<u>UNIQUENESS</u>: IF $h_1^n = h_2^n = f$, THEN $\frac{h_1}{h_2}$ IS CONTINUOUS AND TAKES ON VALUES IN THE FINITE SET OF n^{th} ROOTS OF UNITY. THUS $\frac{h_1}{h_2}$ IS CONSTANT.

FOR ARBITRARY POWERS α,

WE CAN CONSTRUCT A HOLOMORPHIC BRANCH OF f^{α}
IN THE SAME WAY (AS LONG AS f IS NON-VANISHING
IN A SIMPLY CONNECTED DOMAIN U)

TAKE ANY $g \in \mathcal{O}(U)$ SO THAT $f = \exp(g)$.

THEN FOR $\alpha \in \mathbb{C}$, $\exp(\alpha g)$ IS THE HOLOMORPHIC BRANCH
OF f^{α}.

SINCE g IS UNIQUE UP TO INTEGER MULTIPLES OF $2\pi i$

SO BRANCHES OF f^{α} DIFFER BY A MULTIPLICATIVE
CONSTANT OF THE FORM $e^{2\pi i n \alpha}$ $(n \in \mathbb{Z}_+)$.