OWED FROM LAST TIME:

PROOF OF LOCAL NORMAL FORM THM.

THM: $f \in \mathcal{O}(U)$ NONCONSTANT IN A NBHD OF $p \in U$, $f(p) = q$

THEN WE HAVE NEIGHBORHOODS $V \ni p$, $W \ni q$ SO THAT FOR $m \geq 1 \in \mathbb{Z}^+$

$$\begin{array}{ccc} p \in V & \xrightarrow{f} & q \in W \\ \downarrow \varphi & & \downarrow \psi \\ 0 \in \mathbb{D} & \xrightarrow[w \mapsto w^m]{} & 0 \in \mathbb{D} \end{array}$$

Pf/ SINCE $f \in \mathcal{O}(V)$ ~~WE CAN~~ WITH $f(p) = q$,

- WE CAN WRITE $f(z) - q = (z-p)^m g(z)$

WITH $g(p) \neq 0$, $m = \mathrm{ord}(f - q, p)$ $g \in \mathcal{O}(V)$.

WE WANT TO CONSTRUCT AN "m^{TH} ROOT" OF g NEAR p.

SINCE $g(p) \neq 0$, THERE S $r > 0$ SO THAT FOR $D = \mathbb{D}_r(p)$ WE HAVE $g(z) \neq 0$.

THUS $\dfrac{g'(z)}{g(z)}$ IS HOLOMORPHIC ON D AND SO HAS A PRIMITIVE $H \in \mathcal{O}(D)$

SINCE $H' = \dfrac{g'}{g}$

~~SINCE H'~~ OBSERVE $(g e^{-H})' = (g' - gH')e^{-1} = 0$ IN D.

So $g(z) = C e^{H(z)}$ FOR SOME CONST. $C \neq 0$.

ADD A CONSTANT TO $H(z)$ TO GET $g(z) = e^{H(z)}$

LET $\varphi_1(z) = (z-p)\exp\left(\frac{H(z)}{m}\right)$.

NOTE $\varphi_1 \in \mathcal{O}(D)$, $\varphi_1(p) = 0$, $\varphi_1'(p) = \exp\left(\frac{H(z)}{m}\right) \neq 0$

THUS φ_1 IS A BIHOLOMORPHISM FROM $V \to \mathbb{D}_\varepsilon(0)$
ON SOME NBHD V OF p, $V \subset D$

NOW RESCALE:

$$\varphi(z) = \frac{1}{\varepsilon}\varphi_1(z) \qquad \psi(z) = \frac{1}{\varepsilon^m}(z-q)$$

AND WE HAVE

$$\varphi : V \to \mathbb{D} \quad \text{AND} \quad \psi : \mathbb{D}_{\varepsilon^m}(q) \to \mathbb{D}$$

BIHOLOMORPHISMS.

$$\psi(f(z)) = \frac{f(z)-q}{\varepsilon^m} = \frac{(\varphi_1(z))^m}{\varepsilon^m} = (\varphi(z))^m$$

HOLDS FOR $z \in V$.

$\boxtimes$

OPEN MAPPING THM:

LET $U \subset \mathbb{C}$ DOMAIN, $f \in \mathcal{O}(U)$ NON CONSTANT. THEN $f(U)$ OPEN

Pf/ APPLY LOCAL NORMAL FORM:
EVERY $p \in U$ HAS AN OPEN $V(p) \subset U$ WITH $f(V)$ OPEN.

NOW TAKE UNIONS:

$$f(U) = f\left(\bigcup_{p \in U} V(p)\right) = \bigcup_{p\in U} f(V(p)).$$

~~HOLOMORPHIC VERSION~~:

COR: LET $f: U \to \mathbb{C}$ BE INJECTIVE ON A DOMAIN U, $f \in \mathcal{O}(U)$

THEN $f'(z) \neq 0$ FOR ALL $z \in U$, AND

$f: U \to f(U)$ IS A BIHOLOMORPHISM

Pf: IF $f'(p) = 0$ FOR SOME $p \in U$, f IS NOT INJECTIVE AT p (BY LOCAL NORMAL FORM, eg).

~~SINCE~~ $f(U)$ IS OPEN ~~AND~~ BY THE OPEN MAPPING THM, AND f^{-1} IS HOLOMORPHIC BY HOLO. INV. THM

MAXIMUM PRINCIPLE FOR OPEN MAPS:

LET $f: U \to \mathbb{C}$ BE AN OPEN MAP.

(i) $|f|$ CANNOT HAVE A LOCAL MAXIMUM IN U

(ii) $|f|$ CANNOT HAVE A LOCAL MINIMUM IN U IF $f(z)$ IS NONZERO FOR $z \in U$

PROOF: TAKE $p \in U$. SINCE U OPEN, $\exists r > 0$ WITH $D := \mathbb{D}_r(p) \subset U$

(i) SINCE $f(p) \in f(D)$ AND $f(D)$ IS OPEN,
THERE MUST BE $z \in D$ WITH $|f(z)| > f(p)$.
THUS, p IS NOT A MAXIMUM OF $|f|$.
SINCE p ARBITRARY, THERE ARE NONE.

(ii) SIMILARLY, IF $f(p) \neq 0$, THERE IS $w \in D$ WITH $|f(w)| < |f(p)|$, SO NO MINIMUM EITHER.

THM/COR:

IF U IS A BOUNDED DOMAIN WITH $f \in \mathcal{O}(U)$
AND f CONTINUOUS ON $\overline{U}$, THEN

(i) $|f(z)| \leq \sup_{\zeta \in \partial U} |f(\zeta)|$ FOR ALL $z \in U$

(ii) IF $f(z) \neq 0$ FOR ALL $z \in U$ ON U, THEN $|f(z)| \geq \inf_{\zeta \in \partial U} |f(\zeta)|$

IF EQUALITY HOLDS IN EITHER CASE AT SOME z, f IS A CONSTANT

THAT IS "A HOLOMORPHIC FUNCTION ACHIEVES ITS MAXIMUM ON THE BOUNDARY"

Pf/ (i) IF THERE IS A $z \in U$ WITH $|f(z)| > \sup_{\zeta \in \partial U} |f(\zeta)|$

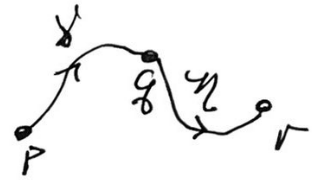
~~THEN |f| HAS A~~
THEN $\sup |f|$ IS ATTAINED ON THE COMPACT SET $\overline{U}$ INSIDE U, ie f HAS A LOCAL MAX.
BUT THEN f IS NOT AN OPEN MAP,
SO IT MUST BE CONSTANT.

(ii) IF f HAS A ZERO ON ∂U, WE ARE DONE.
OTHERWISE $f(z) \neq 0$ FOR ALL $z \in \overline{U}$, AND
WE CAN APPLY (i) TO $\frac{1}{f}$, HOLOMORPHIC IN U.

EX: IF $X \subset \mathbb{R}^n$ IS CONVEX AND $p, q \in X$, THEN ANY TWO CURVES γ_0 AND γ_1 FROM p TO q ARE HOMOTOPIC IN X

- GIVEN $\gamma, \eta : [0,1] \to X$ WITH $\gamma(1) = \eta(0)$

  WE CAN FORM THE PRODUCT (= CONCATENATION)



$$\gamma \cdot \eta(t) = \begin{cases} \gamma(2t) & t \in [0, \tfrac{1}{2}] \\ \eta(2t-1) & t \in [\tfrac{1}{2}, 1] \end{cases}$$

  THIS IS NOT ASSOCIATIVE, BUT

$$\text{Image}(\gamma \cdot \eta \cdot \xi) = \text{Image}\ \gamma \cdot (\eta \cdot \xi)$$

  SO THEY ARE THE SAME AFTER REPARAMETERIZING

- GIVEN $\gamma : [0,1] \to X$, THERE IS A REVERSE CURVE $\gamma^{-} : [0,1] \to X$

  WITH $\gamma^{-}(t) = \gamma(1-t)$.

$$(\eta \cdot \gamma)^{-}(t) = (\gamma^{-} \cdot \eta^{-})(t)$$

- IF $p \in X$ IS A POINT, THE CONSTANT MAP IS DENOTED $\varepsilon_p(t) = p$

THM: HOMOTOPY $\simeq$ IS AN EQUIVALENCE RELATION ON THE SET OF ALL CURVES IN X

PF/ EASY.

$\boxed{\text{THE } \underline{\text{HOMOTOPY CLASS}} \ [\gamma] \text{ OF A CURVE } \gamma \text{ IS THE EQUIVALENCE CLASS OF ALL CURVES } \eta \text{ IN } X \text{ HOMOTOPIC TO } \gamma.}$

OBSERVE $[\eta]\circ[\gamma] = [\eta\circ\gamma]$ IS WELL DEFINED (ASSUMING $\eta(1) = \gamma(0)$).

DEF

LET X BE A TOPOLOGICAL SPACE, $p \in X$.
THEN THE SET OF ALL HOMOTOPY CLASSES OF CLOSED CURVES $\gamma : [0,1] \to X$ WITH $\gamma(0) = \gamma(1) = p$ FORMS A GROUP UNDER $\circ$
THIS GROUP IS THE <u>FUNDAMENTAL GROUP</u> $\pi_1(X, p)$ WITH <u>BASEPOINT p</u>

IN $\pi_1(X, p)$, $[\varepsilon_p]$ IS THE IDENTITY ELEMENT
$[\gamma^-]$ IS THE INVERSE OF $[\gamma]$

SPOZE $f : X \to Y$ IS CONTINUOUS.
IF $H : [0,1]^2 \to X$ IS A HOMOTOPY BETWEEN γ_0 AND γ_1
THEN $f \circ H$ WILL BE A HOMOTOPY BETWEEN $f\circ\gamma_0$ AN $f\circ\gamma_1$

THUS, $f_*([\gamma]) = [f\circ\gamma]$ IS A GROUP HOMEOMORPHISM

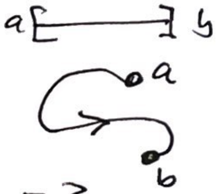
BETWEEN $\pi_1(X)$ AND $\pi_1(Y)$.

HOMOTOPY BASICS

DEFS

A CURVE $\gamma : [a,b] \to \cancel{X}$ IS ← A TOPOLOGICAL SPACE, eg $\mathbb{C}$.

A ~~CLOSED CURVE~~ IF $\gamma(a) = \gamma(b)$.



OTHERWISE, $\gamma(a)$ IS THE INITIAL POINT
$\gamma(b)$ IS THE ENDPOINT

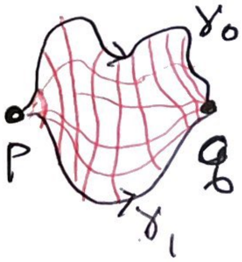
- THE IMAGE $\{x \in X \mid \gamma(t) = x \text{ FOR } t \in [a,b]\}$

WILL BE DENOTED $\{\gamma\}$ (BOOK USES $|\gamma|$)
OR $\text{Image}(\gamma)$.

WE CAN ALWAYS RESCALE SO THAT $[a,b] = [0,1]$

DEF LET $p, q \in X$ WITH

$\gamma_0, \gamma_1 : [0,1]$ BE CURVES FROM p TO q



A HOMOTOPY BETWEEN γ_0 AND γ_1
IS A CONTINUOUS FUNCTION $H : [0,1] \times [0,1] \to X$
SO THAT

- $H(t,0) = \gamma_0(t)$, $H(t,1) = \gamma_1(t)$ FOR ALL $t \in [0,1]$
- $H(0,s) = p$, $H(1,s) = q$ FOR $s \in [0,1]$

IF SUCH A HOMOTOPY EXISTS, γ_0 IS HOMOTOPIC TO γ_1

$\gamma_0 \simeq \gamma_1$

FURTHERMORE, IF $g: Y \to Z$ IS ALSO CONTINUOUS, THEN

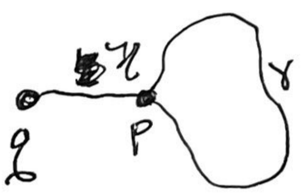
$$(g \circ f)_* = g_* \circ f_*$$

IF $f: X \to Y$ IS A HOMEOMORPHISM, THEN

THEN f_* IS A GROUP ISOMORPHISM FROM $\pi_1(X) \to \pi_1(Y)$ WITH INVERSE $(f^{-1})_*$

$$\boxed{\text{HOMEOMORPHIC SPACES HAVE ISOMORPHIC FUNDAMENTAL GROUPS}}$$

NOTE THAT IF X IS PATH CONNECTED, THE DEPENDENCE ON THE BASEPOINT IS IRRELEVANT.



GIVEN ANY CLOSED PATH γ WITH BASEPOINT P, WE CAN CREATE A CLOSED PATH $\eta \cdot \gamma \cdot \bar{\eta}$, WHERE $\eta(0) = q$, $\eta(1) = p$.

SO ~~THE MAP~~

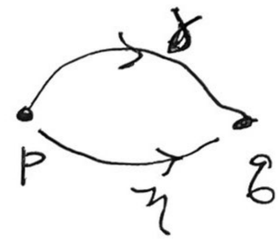
THIS GIVES US AN ISOMORPHISM

$$\pi_1(X, p) \to \pi_1(X, q)$$

DEF: A PATH-CONNECTED TOPOLOGICAL SPACE X IS <u>SIMPLY CONNECTED</u> IF $\pi_1(X, p)$ IS TRIVIAL FOR SOME $p \in X$ (AND HENCE ALL p)

THM: A PATH CONNECTED SPACE X IS ~~SIMPLY~~ SIMPLY CONNECTED IF AND ONLY IF $\Longleftrightarrow$ ANY TWO CURVES WITH SAME INITIAL AND FINAL POINTS ARE HOMOTOPIC

Pf/ LET X BE SIMPLY CONNECTED $\&$ $\gamma \simeq \eta$ ~~BE~~ BE PATHS FROM p TO q.



BUT $\gamma \cdot \eta^{-}$ IS NULL HOMOTOPIC, IE

$$[\gamma]\cdot[\eta]^{-1} = [\gamma\cdot\eta^{-}] = [\varepsilon_p] = id.$$

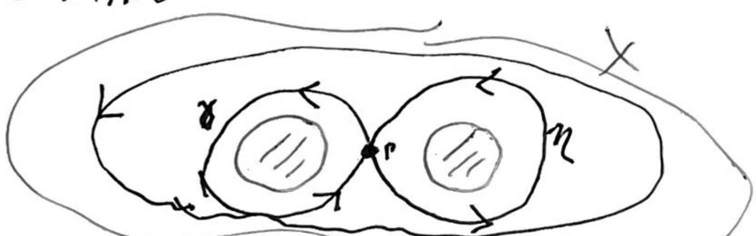
SO $\pi_1(X)$ IS TRIVIAL.

SOMETIMES NO BASEPOINT IS BETTER...

DEF CLOSED CURVES $\gamma_0, \gamma_1 : [0,1] \to X$ ARE FREELY HOMOTOPIC IF THERE IS $H : [0,1]\times[0,1] \to X$ SO THAT

- $H(t,0) = \gamma_0(t)$ $\quad H(t,1) = \gamma_1(t)$ FOR $t \in [0,1]$
- $H(0,s) = H(1,s)$ FOR $s \in [0,1]$.

THAT IS, γ_0 ~~AND~~ CAN BE DEFORMED INTO γ_1 THROUGH A CONTINUOUS FAMILY OF CLOSED ~~CURVE~~ CURVES.



$\eta\cdot\gamma$ IS NOT HOMOTOPIC $\gamma\cdot\eta$ THRU p

BUT IT IS FREELY HOMOTOPIC