DEF

A CONNECTED, NON-EMPTY OPEN SET $U \subset \mathbb{C}$ IS CALLED A <u>DOMAIN</u>.

DEF LET $f \in \mathcal{O}(U)$ WITH f NOT IDENTICALLY ZERO IN A DISK $\mathbb{D}_r(p) \subset U$ BUT $f(p) = 0$.

THE <u>ORDER</u> OF (THE ZERO AT) p IS THE INTEGER

$m = \mathrm{ord}(f, p)$ SO THAT $f(z) = (z-p)^m g(z)$

WITH $g(p) \neq 0$ AND $g \in \mathcal{O}(U)$

IF $\mathrm{ord}(f, p) = 1$, p IS A <u>SIMPLE ZERO</u> OF f

NOTE THIS IS EQUIVALENT TO $f(z) = \sum_{n=0}^{\infty} a_n (z-p)^n = \sum_{n=m}^{\infty} a_n (z-p)^n$

AND $\mathrm{ord}(f, p)$ IS THE SMALLEST NONZERO a_m

SAY HERE

D

<u>LEMMA</u> SPOZE $U \subset \mathbb{C}$ IS A DOMAIN AND $f \in \mathcal{O}(U)$.

IF $f^{-1}(0) = \{z \mid f(z) = 0 \text{ AND } z \in U\}$ HAS AN ACCUMULATION POINT IN U, THEN $f(z) = 0$ FOR ALL $z \in U$

Pf/ LET $E = \{z \in U \mid z$ IS AN ACCUMULATION POINT OF $f^{-1}(0)\}$.

E IS CLOSED IN U AND CONTAINED IN $f^{-1}(0)$ BY CONTINUITY

SPOZE $p \in E$, AND THERE IS SOME DISK $\mathbb{D}_r(p)$ ON WHICH f IS NOT IDENTICALLY 0.

SO WE HAVE THE POWER SERIES $f(z) = (z-p)^m g(z)$

WHERE ~~M IS~~ $m = \operatorname{ord}(f,p)$, $g \in \mathcal{O}(U)$, $g(p) \neq 0$.

AND SO p IS THE ONLY ~~NON~~ ZERO OF f ON ~~THIS~~ A NEIGHBORHOOD OF p, SO $p \notin E$. $\Rightarrow\Leftarrow$

THUS IF $\mathbb{D}_r(p) \subset U$, $f \equiv 0$ ON $\mathbb{D}_r(p)$, SO $\mathbb{D}_r(p) \subset E$,

SO E IS OPEN.

SINCE U IS CONNECTED, $f \equiv 0$ ON U. ∎

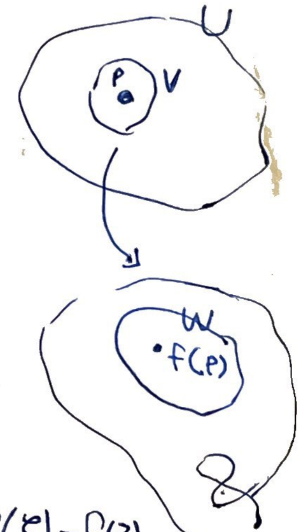
THM: SPOZE $U \subset \mathbb{C}$ IS A DOMAIN AND $f, g \in \mathcal{O}(U)$.

IF THE SET $\{z \in U \mid f(z) = g(z)\}$ HAS AN ACCUMULATION POINT IN U, THEN $f = g$ THROUGHOUT U

Holomorphic Inverse Thm: Let $f \in \mathcal{O}(U)$ with $f'(p) \neq 0$ for some $p \in U$. Then there are open neighborhoods $V \subset U$ of p and $W \subset \mathbb{C}$ of $f(p)$ such that $f : V \to W$ is a bijection.

Further the local inverse $f^{-1} : W \to V$ is holomorphic with

$$\left(f^{-1}\right)'(w) = \frac{1}{f'(f^{-1}(w))} \quad \text{for all } w \in W$$



Pf/ Let $g : U \times U \to \mathbb{C}$ be $g(\zeta, z) = \begin{cases} \dfrac{f(\zeta) - f(z)}{\zeta - z} & z \neq \zeta \\ f'(z) & z = \zeta \end{cases}$

This is continuous at (p,p), so there is $r > 0$ with $g(\zeta, z) \geq \frac{1}{2} g(p,p)$ for $\zeta, z \in \mathbb{D}_r(p)$

That is, for $\zeta, z \in \mathbb{D}_r(p)$, $|f(z) - f(\zeta)| \geq \frac{1}{2} |f'(p)| \, |\zeta - z|$

so f is injective on V. with $|f'(p)| > 0$

Now we show W is open.

Let $z_0 \in V$ and take $s > 0$ so that $\overline{\mathbb{D}_s(z_0)} \subset V$.

Set $\varepsilon = \frac{1}{2} \min\limits_{|z - z_0| = s} |f(z) - f(z_0)|$.

Since f injective, $\varepsilon > 0$.

Pf CONTINUES

SPOZE THERE IS SOME $g \in \mathbb{D}_\varepsilon(f(z_0))$ WITH $g \notin W$.

THEN $h(z) = \dfrac{1}{f(z)-g}$ IS HOLOMORPIC IN V WITH $\sup\limits_{|z-z_0|=\varepsilon} |h(z)| \leq 1/\varepsilon$

USING CAUCHY ESTIMATE $\left|h^{(n)}(z_0)\right| \leq \dfrac{n!}{r^n} \sup\limits_{|z-z_0|=r} |h(z)|$ WITH $n=0$

WE HAVE $|h(z_0)| < 1/\varepsilon$.

BUT $h(z_0) = \dfrac{1}{f(z_0)-g} > 1/\varepsilon$, CONTRADICTION.

THUS $\mathbb{D}_\varepsilon(f(z_0)) \subset W$ ~~THAT IS~~ FOR ANY $z_0 \in V$,

IE W IS OPEN.

FINALLY, TO SEE $f^{-1}: W \to V$ IS HOLOMORPHIC,

TAKE $z, z_0 \in V$ AND LET $w = f(z)$, $w_0 = f(z_0)$

$f'(z_0) \neq 0$ SINCE $|f'(z_0)| \geq \dfrac{|f'(p)|}{2} > 0$.

IF $z \neq z_0$,
$$\frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0} = \frac{z - z_0}{f(z) - f(z_0)}$$

LET $w \to w_0$ SO $z \to z_0$

TO OBTAIN
$$(f^{-1})'(w_0) = \frac{1}{f'(z_0)}$$

REMARK: ~~…~~ ALTHOUGH WE USED f HOLOMORPHIC IN THE PROOF, THE RESULT IS MORE GENERAL:

BROUWER'S INVARIANCE OF DOMAIN THM (1912):

THE IMAGE OF AN OPEN SET IN $\mathbb{R}^n$ UNDER A CONTINUOUS INJECTIVE MAP IS OPEN

DEF: LET $V, W \subset \mathbb{C}$ BE OPEN. THEN $f: V \to W$ IS A BIHOLOMORPHISM IF IT IS BIJECTIVE AND HOLOMORPHIC (AND CONSEQUENTLY $f^{-1}: W \to V$ IS ALSO)

RESTATING THE PREVIOUS THEOREM:

THM

EVERY HOLOMORPHIC FUNCTION IS A LOCAL BIHOLOMORPHISM NEAR POINTS WHERE THE DERIVATIVE IS NONZERO

COMBINING THIS WITH THE ORDER OF A ZERO, WE GET A COMPLETE DESCRIPTION OF THE LOCAL BEHAVIOR:

LOCAL NORMAL FORM LET $f \in \mathcal{O}(U)$ WITH f NON-CONSTANT IN A NEIGHBORHOOD OF $p \in U$. THEN THERE ARE NEIGHBORHOODS $p \in V \subset U$, $\overset{\xi}{\overset{\shortparallel}{f(p)}} \in W \subset \mathbb{C}$

AN INTEGER $m > 0$

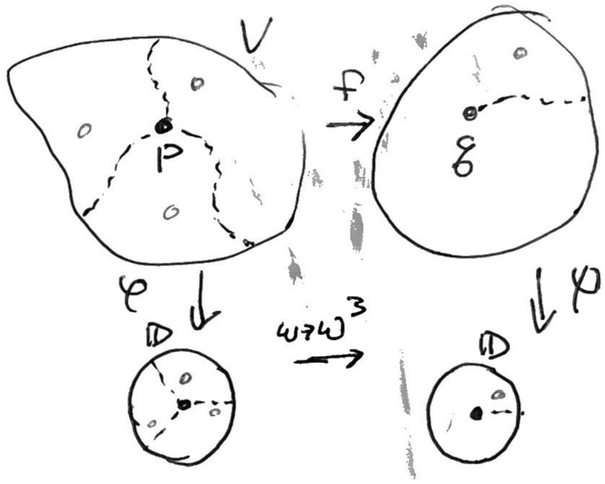
AND BIHOLOMORPHISMS

$\varphi : V \to \mathbb{D}$ $\quad \psi : W \to \mathbb{D}$

SO THAT

$$(\psi \circ f \circ \varphi^{-1})(w) = w^m$$

FOR ALL $w \in \mathbb{D}$



Pf/ SINCE f IS HOLOMORPHIC IN V, WITH $f(p) = \xi$ WE CAN WRITE

$$f(z) - \xi = (z-p)^m g(z)$$

WITH $g(p) \neq 0$, $m = \mathrm{ord}(f-\xi, p)$ $\quad g \in \mathcal{O}(U)$.

WE WANT TO SHOW THAT g HAS A "HOLOMORPHIC m^{th} ROOT" NEAR p.

SINCE $g(p) \neq 0$, WE CAN FIND A SMALL DISK
~~DISK~~ $D = \mathbb{D}_r(p)$ ON WHICH $g(z) \neq 0$, SO

$\frac{g'(z)}{g(z)}$ IS HOLOMORPHIC ON D. THUS, IT

HAS SOME PRIMITIVE $H \in \mathcal{O}(D)$.

SINCE $H' = \frac{g'}{g}$

OBSERVE THAT $(ge^{-H})' = (g' - gH')e^{-H} = 0$ IN D

SO $g(z) = ce^{H(z)}$ FOR SOME CONSTANT $c \neq 0$.

WE CAN ADD A SUITABLE CONSTANT TO H TO ARRANGE THAT $g(z) = e^{H(z)}$ IN D.

NOW LET $\varphi_1(z) = (z-p)\exp\left(\frac{H(z)}{m}\right)$

AND SO $\varphi_1 \in \mathcal{O}(D)$ WITH $\varphi_1(p) = 0$ $\varphi_1'(p) = \exp\left(\frac{H(p)}{m}\right) \neq 0$

φ_1 IS A (~~LOCAL~~ BIHOLOMORPHISM FROM V TO $\mathbb{D}_\varepsilon(0)$ ON SOME $V \subset D$. RESCALE BY ε, i.e.

LETTING. $\varphi(z) = \frac{1}{\varepsilon}\varphi_1(z)$ $\psi(z) = \frac{1}{\varepsilon^m}(z - q_0)$

THEN $\varphi : V \to \mathbb{D}$, $\psi : \mathbb{D}_{\varepsilon^m}(q_0) \to \mathbb{D}$ ARE BIHOLOMORPHIC WITH

$$\psi(f(z)) = \frac{f(z) - q_0}{\varepsilon^m} = \frac{(\varphi_1(z))^m}{\varepsilon^m} = (\varphi(z))^m$$

FOR ALL $z \in V$.

m IS THE LOCAL DEGREE OF f AT p.

MORE FORMALLY:

DEF LET f BE A NON-CONSTANT HOLOMORPHIC FUNCTION IN A NEIGHBORHOOD OF $p \in \mathbb{C}$.

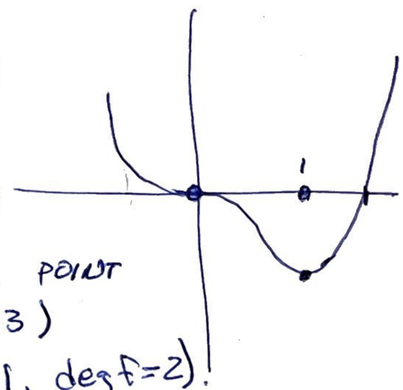
THE ORDER OF p AS A ZERO OF $f - f(p)$ IS THE LOCAL DEGREE OF f AT p

$$\deg(f,p) = \operatorname{ord}(f - f(p), p)$$

IF $\deg(f,p) > 1$ (EQUIVALENTLY, $f'(p) = 0$) THEN p IS A CRITICAL POINT OF f AND $f(p)$ IS THE CORRESPONDING CRITICAL VALUE OF f

$$\deg(f,p) = \operatorname{ord}(f',p) + 1 \qquad \text{IF } p \text{ IS A CRIT. PT.}$$

EXAMPLE: LET $f(z) = z^3(3z-4)$
$= 3z^4 - 4z^3$



THEN $f'(z) = 12z^2(z-1)$, CRITICAL VALUES 0, -1

THAT IS, $z=0$ IS AN ORDER 2 CRITICAL POINT (SO NEAR 0, $\deg f = 3$)

$z=1$ IS ORDER 1 (SO NEAR 1, $\deg f = 2$)

NOTE $f^{-1}(0) = \{0, 4/3\}$ (SOLVING $f(z) = \varepsilon$ GIVES 3 SOL'NS NEAR 0, ONE NEAR 4/3

$f^{-1}(-1) = \left\{1, \dfrac{-1 \pm i\sqrt{2}}{3}\right\}$ SOLVING $f(z) = \varepsilon - 1$ GIVES TWO NEAR 1, ONE EACH NEAR OTHERS

THE OPEN MAPPING THEOREM:

SPOZE $U \subset \mathbb{C}$ IS A DOMAIN, $f \in \mathcal{O}(U)$ NON-CONSTANT. THEN $f(U)$ IS OPEN

Pf/ BY LOCAL NORMAL FORM, EVERY $p \in U$ HAS AN OPEN NBHD. $V(p) \subset U$ SUCH THAT $f(V(p))$ IS OPEN. THUS $f(U) = f\left(\bigcup_{p \in U} V(p)\right) = \bigcup_{p \in U} f(V(p))$ ☒

COR LET $f: U \to \mathbb{C}$ BE INJECTIVE ON A DOMAIN U, WITH $f \in \mathcal{O}(U)$

THEN $f'(z) \neq 0$ FOR ALL $z \in U$ AND $f: U \to f(U)$ IS A BIHOLOMORPHISM

Pf/ IF $f'(p) = 0$ FOR SOME $p \in U \Rightarrow f$ IS NOT INJECTIVE IN A NBHD OF U.
$f(U)$ OPEN BY ABOVE, AND f^{-1} IS HOLOMORPHIC BY THE HOLOMORPHIC INVERSE THM.

REMARK THE CONVERSE IS FALSE.
CONSIDER $f(z) = e^z$. $f'(z) \neq 0$ FOR ALL $z \in \mathbb{C}$ BUT e^z IS INFINITE TO ONE.

REMARK THE COROLLAREY FAILS FOR SMOOTH MAPS $f: \mathbb{R}^2 \to \mathbb{R}^2$. CONSIDER $f(x,y) = (x^3, y)$ WHICH IS C^∞, BUT EVERY POINT OF THE FORM $(0,y)$ IS CRITICAL, AND THE INVERSE

$$f^{-1}(x,y) = (\sqrt[3]{x}, y)$$ IS NOT DIFF. AT $x = 0$.

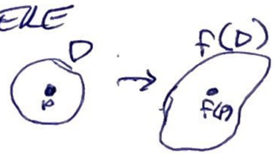
THM: THE MAXIMUM PRINCIPLE FOR OPEN MAPS

SPOZE $f: U \to \mathbb{C}$ IS AN OPEN MAP. THEN

i) $|f|$ CANNOT HAVE A LOCAL MAXIMUM IN U

ii) $|f|$ CANNOT HAVE A LOCAL MINIMUM IN U
IF $f(z) \neq 0$ FOR ALL $z \in U$

Pf/ LET $p \in U$, AND LET $D = \mathbb{D}_r(p)$ WITH $\mathbb{D}_r(p) \subset U$.
SINCE $f(D)$ IS OPEN AND CONTAINS $f(p)$, THERE IS A POINT $z \in D$ SO THAT $|f(z)| > |f(p)|$



THUS, $|f|$ HAS NO MAXIMUM ON D.
SIMILARLY, IF $f(p) \neq 0$, THERE IS $w \in D$ WITH $|f(w)| < |f(p)|$,
SO THERE CAN BE NO MINIMUM.

IF WE ASSUME $f \in \mathcal{O}(U)$ WITH U A BOUNDED DOMAIN:

THM (HOLOMORPHIC MAXIMUM PRINCIPLE)

SPOZE U IS A BOUNDED DOMAIN WITH $f: \overline{U} \to \mathbb{C}$ CONTINUOUS AND $f \in \mathcal{O}(U)$, THEN

(i) $|f(z)| \leq \sup_{\zeta \in \partial U} |f(\zeta)|$ FOR ALL $z \in U$

(ii) $|f(z)| \geq \inf_{\zeta \in \partial U} |f(\zeta)|$ FOR ALL $z \in U$ (PROVIDED $f \neq 0$ ON U)

IF EQUALITY HOLDS IN EITHER CASE AT SOME z, f IS CONSTANT

ie "A HOLOMORPHIC f ACHIEVES ITS MAXIMUM ON THE BOUNDARY"

Pf/ (i) IF $|f(z)| \geq \sup_{\zeta \in \partial U} |f(\zeta)|$ FOR $z \in U$,

THEN $\sup|f|$ ON THE COMPACT SET $\overline{U}$ IS ATTAINED INSIDE U, AND HENCE f HAS A LOCAL MAXIMUM. BUT THEN f CANNOT BE AN OPEN MAP, SO IT IS CONSTANT.

(ii) IF f HAS A ZERO ON ∂U, WE ARE DONE. OTHERWISE $f \neq 0$ ON $\overline{U}$, SO WE CAN APPLY (i) TO $1/f$ WHICH IS HOLOMORPHIC ON U