LAST TIME:

THM: f CONTINUOUS, $f: U \to \mathbb{C}$

f HAS A PRIMITIVE IN U $\iff$ $\int_\gamma f(z)\,dz = 0$ FOR EVERY CLOSED $\gamma \subset U$

GOURSAT'S THM

IF $f \in \mathcal{O}(U)$, THEN FOR EVERY CLOSED TRIANGLE $T \subset U$, WE HAVE $\int_{\partial T} f(z)\,dz = 0$

COR LET $f \in \mathcal{O}(D)$ WHERE D IS AN OPEN DISK IN $\mathbb{C}$. THEN f HAS A PRIMITIVE IN D.

THIS IMPLIES

THM (CAUCHY'S THM IN A DISK) (1825): LET D BE AS ABOVE. THEN FOR EVERY CLOSED γ IN D, $\int_\gamma f(z)dz = 0$

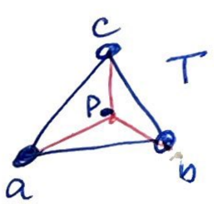
(**) [NEED FOR PF OF CAUCHY IN DISK]

REMARK

IN FACT, CAUCHY'S THM HOLDS IF f IS CONTINUOUS IN D AND HOLOMORPHIC IN $D \setminus \{p\}$ FOR SOME $p \in D$.

TO SEE THIS, IT SUFFICES TO SHOW $\int_{\partial T} f(z)dz = 0$ FOR EVERY CLOSED TRIANGLE $T \subset D$. • IF $T \subset D \setminus \{p\}$, THEN WE ALREADY HAVE THE RESULT, SO ASSUME $p \in T$.

NOW, FIRST CONSIDER $p \in \partial T$. WE CAN FIND A TRIANGLE T' ARBITRARILY CLOSE TO T WITH $p \notin T'$, SO $\int_{\partial T'} f(z)dz = 0$.

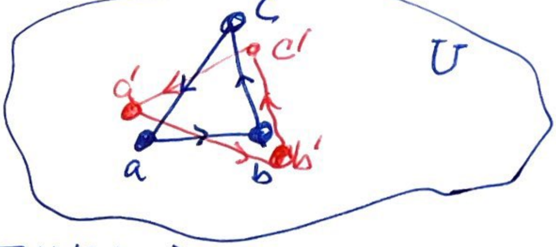


(SEE LEMMA IN NOTES OR TEXT)

THE VALUE OF $\int_T f(z)dz$ DEPENDS CONTINUOUSLY ON THE VERTICES, SO $\left| \int_T f(z)dz - \int_{T'} f(z)dz \right| < \varepsilon$ FOR ANY ε, IE $\int_T f(z)dz = 0$.

IF $p \in$ INTERIOR T, WE CAN EXPRESS

$$\int_{\partial T} f(z)dz = \int_{\partial \Delta apb} f(z)dz + \int_{\partial \Delta bpc} f(z)dz + \int_{\partial \Delta cpa} f(z)dz$$



LEMMA (CONTINUOUS DEPENDENCE ON VERTICES)

LET $f: U \to \mathbb{C}$ BE CONTINUOUS, AND $T = \Delta abc$ A CLOSED TRIANGLE IN $\mathbb{C}$.
THEN FOR EVERY $\varepsilon > 0$ THERE IS A $\delta > 0$ SUCH THAT IF $|a-a'| < \delta$, $|b-b'| < \delta$, $|c-c'| < \delta$ THEN $T' = \Delta a'b'c' \subset U$ AND

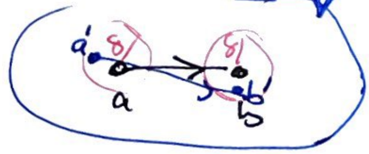
$$\left| \int_{\partial T} f(z)\,dz - \int_{\partial T'} f(z)\,dz \right| < \varepsilon$$

— SKIP IN CLASS

PROOF IS STRAIGHTFORWARD, BUT A LITTLE TEDIOUS

Pf/ SINCE THE INTEGRAL ALONG ∂T IS JUST THE SUM OF INTEGRALS ALONG SIDES, WE JUST SHOW IT FOR ORIENTED SEGMENTS. $[a,b]$ AND $[a',b']$.

FIX $[a,b] \subset U$, LET $V \subset U$ BE OPEN WITH $\overline{V}$ A COMPACT SUBSET OF U



BY UNIFORM CONTINUITY OF f, FOR $\varepsilon > 0$ THERE IS $0 < \delta < \varepsilon$ SO THAT $|f(z) - f(w)| < \varepsilon$ FOR ALL $z, w \in V$ WITH $|z-w| < \delta$.

ALSO, $[a',b'] \subset V$ IF $|a-a'| < \delta$, $|b-b'| < \delta$

SKIP IN CLASS

LET $\gamma(t) = (1-t)a + tb$, $\gamma_1 = (1-t)a' + tb'$

AND SO $|\gamma(t) - \gamma_1(t)| \le (1-t)|a-a'| + t|b-b'| < (1-t)\delta + t\delta \le \delta$

AND $|\gamma'(t) - \gamma_1'(t)| \le |(b-a)-(b'-a')| \le |b-b'| + |a-a'| < 2\delta$.

FOR $0 \le t \le 1$

INTEGRATING,

$$\left| \int_{[a,b]} f(z)\,dz - \int_{[a',b']} f(z)\,dz \right| = \left| \int_0^1 f(\gamma(t))\gamma'(t) - f(\gamma_1(t))\gamma_1'(t)\,dt \right|$$

$$= \int_0^1 \big(f(\gamma(t)) - f(\gamma_1(t))\big)\gamma'(t) + f(\gamma(t))\big(\gamma'(t) - \gamma_1'(t)\big)\,dt$$

$$\le |b-a| \int_0^1 |f(\gamma(t)) - f(\gamma_1(t))|\,dt + 2\delta \int_0^1 f(\gamma_1(t))\,dt$$

$$\le |b-a|\varepsilon + 2\delta \sup_{z\in V} |f(z)| \le |b-a| + 2\varepsilon \sup_{z \in V} |f(z)|.$$

$\boxtimes$

DEF: A FUNCTION $F \in \mathcal{O}(U)$ IS A PRIMITIVE OF A CONTINUOUS FUNCTION $f: U \to \mathbb{C}$ IF $F'(z) = f(z)$ FOR ALL $z \in U$

(AN ANTIDERIVATIVE IN CALCULUS)

SEE ELSEWHERE

GIVEN A PIECEWISE C' CURVE $\gamma: [0,1] \to \mathbb{C}$, AND A PRIMITIVE F OF f

BY THE CHAIN RULE,

$$(F \circ \gamma)'(t) = F'(\gamma(t))\gamma'(t) \quad \text{FOR ALL BUT FINITELY MANY } t.$$

THM

CAUCHY INTEGRAL FORMULA IN A DISK: LET $D \subset \mathbb{C}$ BE AN OPEN DISK, AND $f \in \mathcal{O}(D)$.

IF $\overline{\mathbb{D}_r(p)} \subset D$, THEN

$$f(z) = \frac{1}{2\pi i} \int_{\mathbb{T}_r(p)} \frac{f(\zeta)\, dz}{\zeta - z} \qquad \text{FOR } z \in \mathbb{D}_r(p)$$

RECALL $\mathbb{T}_r(p) = \partial \mathbb{D}_r(p)$

IN OTHER WORDS, THE VALUES OF f ON THE CIRCLE $\mathbb{T}_r(p)$ DETERMINE THE VALUES ON THE INTERIOR.

NOTE THAT THIS IS RELATED (IN A WAY WE WILL MAKE PRECISE LATER) TO THE UNSURPRISING FACT THAT GIVEN $f : \mathbb{R} \to \mathbb{R}$ IS "NICE" (IE ANALYTIC) THERE IS ONLY ONE WAY TO EXTEND IT TO A HOLOMORPHIC ~~$f: \mathbb{C} \to \mathbb{C}$~~ FUNCTION DEFINED IN $\mathbb{C}$ (NOT NECESSARILY ENTIRE)

BECAUSE CIRCLES & LINES ARE "THE SAME".

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Pf of Cauchy in Disk:

Fix $z \in \mathbb{D}_r(p)$ and let $g : D \to \mathbb{C}$ be

$$g(\zeta) = \begin{cases} \dfrac{f(\zeta) - f(z)}{\zeta - z} & \zeta \neq z \\ f'(z) & \zeta = z \end{cases}$$

g is continuous and holomorphic in $D \setminus \{z\}$.

So (using the (**) remark on 2) $\int_{\mathbb{T}_r(p)} g(\zeta)\,d\zeta = 0$.

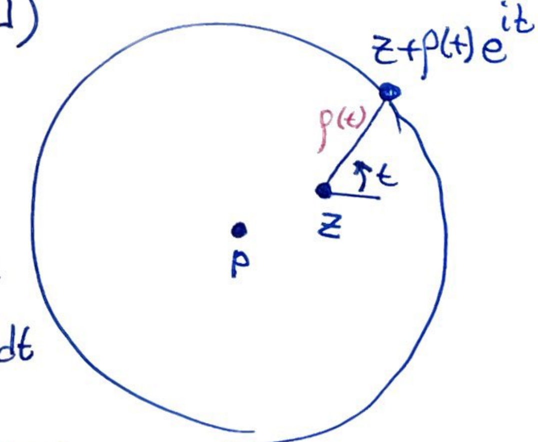
Hence

$$\frac{1}{2\pi i}\int_{\mathbb{T}_r(p)} \frac{f(\zeta)}{z-\zeta}\,d\zeta = f(z)\cdot\frac{1}{2\pi i}\underbrace{\int_{\mathbb{T}_r(p)}\frac{1}{\zeta - z}\,d\zeta}_{\text{want this to be } 2\pi i}$$

Now parameterize $\mathbb{T}_r(p)$ by $z + \rho(t)e^{it} = \gamma$ where $\rho(t) \in \mathbb{R}^+$ so that ($t \in [0, 2\pi]$)

$$|z + \rho(t)e^{it} - p| = r$$

($t \mapsto \rho(t)$ is cont. differentiable)



So

$$\int_{\mathbb{T}_r}\frac{1}{\zeta - z}\,d\zeta = \int_0^{2\pi}\frac{(\rho'(t) + i\rho(t))e^{it}}{\rho(t)}\,dt$$

$$= \int_0^{2\pi}\frac{\rho'(t)}{\rho(t)}\,dt + 2\pi i$$

$$= \underbrace{\log(\rho(2\pi)) - \log(\rho(0))}_{0 \text{ since } \rho(2\pi) = \rho(0)} + 2\pi i = 2\pi i$$

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NOW WE HAVE

HOLOMORPHIC $\Rightarrow$ COMPLEX ANALYTIC

THM: IF $f \in \mathcal{O}(U)$, THEN IN ~~AN~~ EVERY DISK $\mathbb{D}_r(p) \subset U$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-p)^n$$

WHERE

$$a_n = \frac{f^{(n)}(p)}{n!} = \frac{1}{2\pi i} \int_{\mathbb{T}_s(p)} \frac{f(\zeta)}{(\zeta-p)^{n+1}}\, d\zeta$$

FOR ANY $0 < s < r$

IMMEDIATE COR: IF $f \in \mathcal{O}(U)$, THEN $f' \in \mathcal{O}(U)$ (AND HENCE $f^{(k)} \in \mathcal{O}(U)$ FOR $k \geq 1$)

PF/ FIX s WITH $0 < s < r$ AND SOME $z \in \mathbb{D}_s(p)$.

IF $\zeta \in \mathbb{T}_s(p)$, $\displaystyle \frac{1}{\zeta - z} = \frac{1}{(\zeta-p)\left(1 - \frac{z-p}{\zeta-p}\right)} = \frac{1}{\zeta-p} \cdot \sum_{n=0}^{\infty} \left(\frac{z-p}{\zeta-p}\right)^n$

CONV UNIF. IN ζ

TERM-BY TERM INTEGRATION WRT ζ

$$f(z) = \frac{1}{2\pi i} \int_{\mathbb{T}_s(p)} \sum_{n=0}^{\infty} \frac{f(\zeta)(z-p)^n}{(\zeta-p)^{n+1}}\, d\zeta = \sum_{n=0}^{\infty} a_n (z-p)^n$$

WHICH HOLDS FOR $z \in \mathbb{D}_s(p)$. ☒

~~COR: IF $f \in \mathcal{O}(U)$, $f' \in \mathcal{O}(U)$ AND HENCE $f^{(k)} \in \mathcal{O}(U)$ FOR $k \geq 1$.~~

~~DUH~~

SINCE THE INTEGRAL DEPENDS CONTINUOUSLY ON VERTICES AND $\int_{\partial T} f(z)dz = 0$ WE GET THE RESULT

THE CONVERSE TO GOURSAT'S THM:

MORERA'S THM (1886) LET $f: U \to \mathbb{C}$ BE CONTINUOUS, WITH $\int_{\partial T} f(z)\,dz = 0$ FOR EVERY CLOSED TRIANGLE $T \subset U$. THEN $f \in \mathcal{O}(U)$

PF/ LET $D \subset U$ BE A DISK. THEN WE KNOW f HAS A PRIMITIVE F IN D WITH $F \in \mathcal{O}(D)$.

BUT THEN $f = F'$ IS ALSO HOLOMORPHIC.

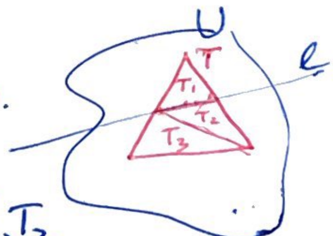
SINCE $f \in \mathcal{O}(D)$ FOR EVERY DISK $D \subset U$,

$f \in \mathcal{O}(U)$ ∎

<u>REMARK</u> MORERA'S THEOREM (LIKE GOURSAT) ALSO HOLDS FOR OTHER FAMILIES OF CLOSED CURVES, LIKE CLOSED RECTANGLES OR SQUARES.

<u>THM:</u> (LINES ARE REMOVABLE)

LET $U \subset \mathbb{C}$ BE OPEN WITH $f: U \to \mathbb{C}$ CONTINUOUS.
LET ℓ BE A STRAIGHT LINE WITH $\ell \cap U \neq \emptyset$.
SPOZE $f \in \mathcal{O}(U \setminus \ell)$. THEN $f \in \mathcal{O}(U)$.



PF/ WE SHOW THAT FOR ANY $T \subset U$, $\int_{\partial T} f(z)\,dz = 0$.

IF $T \cap \ell = \emptyset$, WE ARE DONE.
OTHERWISE DECOMPOSE T INTO (AT MOST) T_1, T_2, T_3 SO THAT THE INTERIOR OF T_j IS DISJOINT FROM ℓ.
FOR SUCH T_j WE CAN MOVE TO T_j' SO THAT $T_j' \cap \ell = \emptyset$.
SINCE THE INTEGRAL DEPENDS CONTINUOUSLY ON VERTICES AND $\int_{\partial T_j'} f(z)\,dz = 0$ WE GET THE RESULT

THM: (CAUCHY ESTIMATES, 1835) LET f BE CONTINUOUS ON $\overline{\mathbb{D}_r(p)}$, $f \in \mathcal{O}(\mathbb{D}_r(p))$. THEN

$$|f^{(n)}(p)| \leq \frac{n!}{r^n} \cdot \sup_{|z-p|=r} |f(z)| \qquad \forall\, n \geq 0$$

Pf/ LET $0 < s < r$ AND WRITE f AS A POWER SERIES $f(z) = \sum_{n=0}^{\infty} a_n (z-p)^n$ IN $\mathbb{D}_s(p)$

THEN SINCE $a_n = \frac{f^{(n)}(p)}{n!} = \frac{1}{2\pi i} \int_{\mathbb{T}_s(p)} \frac{f(\zeta)}{(\zeta-p)^{n+1}}\, d\zeta$

TAKING ABS VALUES GIVES

$$|f^{(n)}(p)| \leq \frac{n!}{2\pi} \left| \int_{\mathbb{T}_s(p)} \frac{f(\zeta)}{(\zeta-p)^{n+1}}\, d\zeta \right|$$

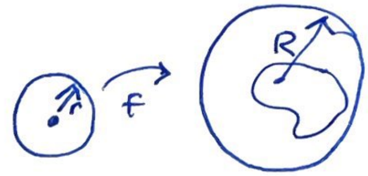
$$\leq \frac{n!}{2\pi} \cdot \sup_{|z-p|=s} \frac{|f(z)|}{|z-p|^{n+1}} \cdot 2\pi s \qquad \text{(BY M-L)}$$

$$= \frac{n!}{s^n} \sup_{|z-p|=s} |f(z)|$$

NOW LET $s \to r$. ∎

COR: LET f BE HOLOMORPHIC WITH $f(\mathbb{D}_r(p)) \subset \mathbb{D}_R(q)$

THEN $|f'(p)| \leq R/r$.



IN PARTICULAR, IF $f \in \mathcal{O}(\mathbb{D})$ WITH $f(\mathbb{D}) \subset \mathbb{D}$, $|f'(0)| \leq 1$. (SIMPLEST SCHWARZ LEMMA)

Pf/ LET $0 < s < r$, LOOK AT $g(z) = f(z) - q$:

$g : \mathbb{D}_r(p) \to \mathbb{D}_R(p)$

$$|f'(p)| = |g'(p)| \leq \frac{1}{s} \sup_{|z-p|=s} |g(z)|$$

$$\leq \frac{R}{s}.$$

LET $s \to r$,

THM (LIOUVILLE'S THM 1847)

EVERY BOUNDED ENTIRE FUNCTION IS CONSTANT

Pf/ LET $f \in \mathcal{O}(\mathbb{C})$ WITH $|f(z)| < M \ \forall z \in \mathbb{C}$.

THEN $f(\mathbb{D}_r(p)) \subset \mathbb{D}_M(0)$ FOR ANY r, p.

SO $|f'(p)| \leq \frac{M}{r}$.

LET $r \to \infty$ TO SEE $f'(p) = 0$ FOR ANY $p \in \mathbb{C}$

THAT IS, $f(z)$ IS CONSTANT

$\boxtimes$

COR OF LIOUVILLE:

THE FUNDAMENTAL THEOREM OF ALGEBRA:
IF $P: \mathbb{C} \to \mathbb{C}$ IS A POLYNOMIAL, OF DEGREE d
THEN $P(z) = a(z-\rho_1)(z-\rho_2)\cdots(z-\rho_d)$
FOR $\rho_i \in \mathbb{C}$ (NOT NECESS DISTINCT

Pf. OBSERVE P IS ENTIRE AND $\lim_{z\to\infty} P(z) = \infty$

IF $P(z) \neq 0$ FOR ALL z, THEN $\frac{1}{P(z)}$ IS ENTIRE

WITH $\lim_{z\to\infty} \frac{1}{P(z)} = 0$.

THUS THERE IS $R > 0$ SO THAT $\left|\frac{1}{P(z)}\right| < 1$

FOR $|z| > R$. THUS $f(z) = \frac{1}{P(z)}$ IS BOUNDED

ON $\overline{\mathbb{D}_R(0)}$ AND HENCE ON $\mathbb{C}$.

THUS f IS CONSTANT.

SO $\exists \rho_1$ SO THAT $P(\rho_1) = 0$.

WRITE $P(z) = (z-\rho_1)P_1(z)$ WHERE $P_1(z)$

A POLYNOMIAL WITH $\deg P_1(z) = d-1$

REPEAT UNTIL $P_d(z)$, WHICH IS CONSTANT.

(4

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IF TIME:

THM: LET $U \subset \mathbb{C}$ OPEN, $\varphi : U \times [a,b] \to \mathbb{C}$ CONTINUOUS AND SUCH THAT FOR EACH $t \in [a,b]$, $z \mapsto \varphi(z,t)$ IS HOLOMORPHIC WITH DERIVATIVE $\varphi'(z,t)$.

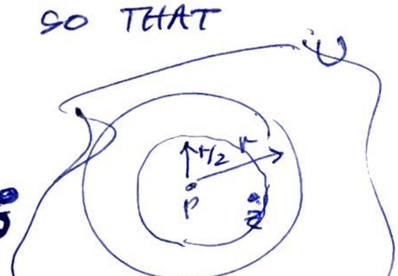
THEN $f(z) = \int_a^b \varphi(z,t)\,dt$ IS HOLOMORPHIC ON U

WITH $f'(z) = \int_a^b \varphi'(z,t)\,dt$ FOR $z \in U$.

IN OTHER WORDS, WE CAN DIFFERENTIATE UNDER THE INTEGRAL SIGN.

REMARK THIS HOLDS FOR DERIVATIVES, IE $f^{(n)}(z) = \int_a^b \varphi^{(n)}(z,t)\,dt$

Pf/ FIX $p \in U$, AND CHOOSE $r > 0$ SO THAT $\overline{\mathbb{D}_r(p)} \subset U$, WITH $z \in \mathbb{D}_{r/2}(p)$



THEN $\varphi(z,t) - \varphi(p,t) = \frac{1}{2\pi i}\int_{\mathbb{T}_r(p)} \varphi(\zeta,t)\cdot\left(\frac{1}{\zeta - z} - \frac{1}{\zeta - p}\right) d\zeta$

THUS $\frac{\varphi(z,t) - \varphi(p,t)}{z-p} = \frac{1}{2\pi i}\int_{\mathbb{T}_r(p)} \frac{\varphi(\zeta,t)}{(\zeta - z)(\zeta - p)}\,d\zeta$

FOR $\zeta \in \mathbb{T}_r(p)$, $|\zeta - z| > r/2$, SO

$$\left|\frac{\varphi(z,t) - \varphi(p,t)}{z-p}\right| \le \frac{1}{2\pi}\cdot M\cdot\frac{2}{r^2}\cdot 2\pi r = \frac{2M}{r},$$ WHERE $M = \sup_{\overline{\mathbb{D}_r(p)} \times [a,b]} |\varphi|$

NOW CHOOSE ANY $\{z_n\}$ IN $U \setminus \{p\}$ WITH $z_n \to p$,

AND LET $g_n(t) = \dfrac{\varphi(z_n,t) - \varphi(p,t)}{z_n - t}$. $\{g_n\}$ IS A SEQ. OF CONTINUOUS FUNCTIONS CONVERGING (POINTWISE) TO $\varphi'(p,t)$ AND $|g_n(t)| < \frac{2M}{r}$ FOR n LARGE ENOUGH.

SO $t \mapsto \varphi'(p,t)$ IS INTEGRABLE ON $[a,b]$ (BY LEBESGUE DOMINATED CONVERGENCE)

AND
$$\lim_{n\to\infty} \frac{f(z_n) - f(p)}{z_n - p} = \lim_{n\to\infty} \int_a^b g_n(t)\,dt = \int_a^b \varphi'(p,t)\,dt.$$

THUS $f'(p) = \int_a^b \varphi'(p,t)\,dt$

COR LET $\gamma : [0,1] \to \mathbb{C}$ BE PIECEWISE C^1, AND $g : \mathrm{Image}(\gamma) \to \mathbb{C}$ BE CONTINUOUS. THEN FOR INTEGER $n \geq 1$,
$$f(z) = \int_\gamma \frac{g(\zeta)}{(\zeta - z)^n}\,d\zeta$$
IS HOLOMORPHIC ON $\mathbb{C} \setminus \mathrm{IMAGE}(\gamma)$

AND
$$f'(z) = n \int_\gamma \frac{g(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta$$
FOR $z \in \mathbb{C} \setminus \mathrm{IMAGE}(\gamma)$

PF/ JUST APPLY THM TO $\varphi(z,t) = \dfrac{g(\gamma(t))\,\gamma'(t)}{(\gamma(t) - z)^n}$.

CAUCHY INTEGRAL FORMULA FOR DERIVATIVES

OBSERVE THAT IF $f \in \mathcal{O}(U)$ WITH $\overline{\mathbb{D}}_r(P) \subset U$

$\frac{1}{2\pi i}\int_{\mathbb{T}_r(P)} \frac{f(\zeta)}{\zeta - z}\,dz$ IS HOLOMORPHIC ON $\mathbb{C} \setminus \mathbb{T}_r(P)$.

DIFFERENTIATING UNDER THE INTEGRAL SIGN GIVES

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{\mathbb{T}_r(P)} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta$$