MAT 536 — 1/30

LAST TIME, WE DEFINED $\int_\gamma f(z)dz$ AS THE PATH INTEGRAL OVER A PARAMETERIZED CURVE $\gamma:[a,b]\to\mathbb{C}$, IE $\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$

WHERE WE CAN JUST COMPUTE THE RHS AS A PAIR OF (REAL) PATH INTEGRALS USING THE C-R EQUATIONS TO WRITE IN TERMS OF dx & dy. BUT THAT'S ANNOYING.

AS IN CALCULUS, WE'D LIKE A BETTER WAY.

DEF: A FUNCTION $F\in\mathcal{O}(U)$ IS A <u>PRIMITIVE</u> OF A CONTINUOUS FUNCTION $f:U\to\mathbb{C}$ IF
$$F'(z)=f(z) \quad \text{FOR ALL } z\in U$$

IN OTHERWORDS, IT IS AN "ANTI DERIVATIVE"

NOW SPOZE F IS A PRIMITIVE OF f, AND $\gamma:[0,1]\to U$ IS A PIECEWISE C^1 CURVE. OBSERVE THAT $(F\circ\gamma)'(t)=F'(\gamma(t))\gamma'(t)$ FOR ALL BUT FINITELY MANY $t\in[0,1]$ BY THE CHAIN RULE. BUT THIS IS PIECEWISE CONTINUOUS (WITH AT WORST JUMP DISCONTINUITIES), SO BY THE FUND. THM. OF CALCULUS,

$$\int_\gamma f(z)dz = \int_0^1 f(\gamma(t))\gamma'(t)dt = \int_0^1 F'(\gamma(t))\,\gamma'(t)\,dt = \int_0^1 (F\circ\gamma)'(t)dt$$
$$= F(\gamma(1)) - F(\gamma(0))$$

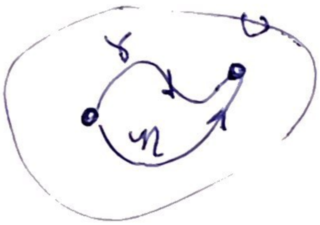
THM: A CONTINUOUS FUNCTION $f : U \to \mathbb{C}$ HAS A PRIMITIVE IN U $\iff$ $\int_\gamma f(z)\,dz = 0$ FOR EVERY CLOSED $\gamma \in U$

Pf $\Rightarrow$/ SPOZE f HAS A PRIMITIVE. APPLY THE FTC

$$\int_\gamma f(z)\,dz = F(\gamma(1)) - F(\gamma(0)) = 0 \quad \text{SINCE } \gamma \text{ CLOSED.}$$

$\Leftarrow$/ CONVERSELY, SPOZE $\int_\gamma f = 0$ ON EVERY CLOSED $\gamma \in U$.

WE CAN RESTRICT TO THE CASE OF U PATH CONNECTED (IF NOT, APPLY ON EACH COMPONENT)



LET γ, η BE TWO CURVES IN U
WITH $\gamma(0) = \eta(0)$, $\gamma(1) = \eta(1)$,
SO $\gamma \cdot \eta^{-1}$ IS CLOSED.

(*)

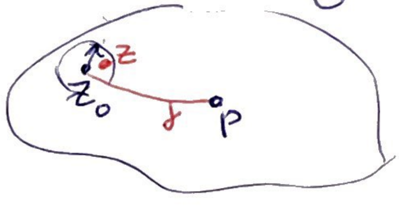
THEN $\int_\gamma f(\zeta)\,d\zeta - \int_\eta f(\zeta)\,d\zeta = \int_{\gamma\circ\eta^{-1}} f(\zeta)\,d\zeta \underset{\text{BY ASSUMPTION}}{=} 0$

FIX SOME $p \in U$, AND FOR ANY $z \in U$, LET γ_z BE A ~~PATH~~ CURVE FROM p TO z IN U.

DEFINE $F(z) = \int_{\gamma_z} f(\zeta)\,d\zeta$

BY (*), F IS INDEPENDENT OF THE CHOICE OF γ_z SO IT IS A WELL-DEFINED FUNCTION $F : U \to \mathbb{C}$.

TO SEE IT IS A PRIMITIVE, FIX $z_0 \in U$, AND ε SO THAT $\mathbb{D}_\varepsilon(z_0) \subset U$



AND LET γ BE A CURVE FROM p TO z.

THEN $$F(z)-F(z_0) = \int_{\gamma\cdot[z_0,z]} f(\zeta)d\zeta - \int_\gamma f(\zeta)d\zeta = \int_{[z_0,z]} f(\zeta)d\zeta$$

NOW, IF $z \neq z_0$,

$$\frac{F(z)-F(z_0)}{z-z_0} - f(z_0) = \frac{1}{z-z_0}\int_{[z_0,z]} \big(f(\zeta)-f(z_0)\big)\,d\zeta$$

SINCE f IS CONTINUOUS AT z_0, FOR $\varepsilon > 0$. THERE IS A $0<\delta<r$ SO THAT $|f(\zeta)-f(z_0)|<\varepsilon$ WHEN $|\zeta-z_0|<\delta$. OBSERVE THAT $|z-z_0|<\delta \Rightarrow |\zeta-z_0|<\delta$ FOR $\zeta\in[z_0,z]$. APPLY THE ML INEQUALITY:

$$\left|\frac{F(z)-F(z_0)}{z-z_0} - f(z_0)\right| \leq \frac{1}{|z-z_0|}\cdot\varepsilon\cdot \text{LENGTH}[z_0,z] = \varepsilon$$

AS LONG AS $0<|z-z_0|<\delta$.

THUS $F'(z_0)$ EXISTS AND $F'(z_0)=f(z_0)$.

SINCE z_0 WAS ARBITRARY, F IS A PRIMITIVE OF f IN U.

EX: NOTE FOR $n \neq -1$, $n\in\mathbb{Z}$, $f(z)=z^n$ HAS A PRIMITIVE

$$F(z) = \frac{z^{n+1}}{n+1}$$ ~~ON ALL OF~~

THUS $$\int_\gamma z^n\,dz = 0 \text{ IF } \begin{cases} \gamma\in\mathbb{C} \text{ CLOSED}, & n\geq 0 \\ \gamma\in\mathbb{C}-\{0\} \text{ CLOSED}, & n<-1. \end{cases}$$

BUT IF $n=-1$, FOR ANY $r>0$: PARAMETRIZE $\gamma(t)=re^{it}$

$$\int_{\Pi_r(0)} \frac{1}{z}\,dz = \int_0^{2\pi}\frac{1}{re^{it}}\cdot \underbrace{ire^{it}}_{\gamma'}\,dt = 2\pi i \neq 0$$

SO $z\to\frac{1}{z}$ HAS NO PRIMITIVE IN $\mathbb{C}-\{0\}$.

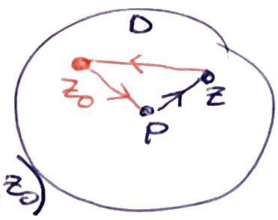
OUR GOAL NOW IS TO SHOW THAT IF $f \in \mathcal{O}(\mathbb{D})$, THEN f HAS A PRIMITIVE IN $\mathbb{D}$. (ACTUALLY, $D_r(p)$ BUT EQUIV.)

WE CAN SNEAK UP ON THIS USING TRIANGLES (OR RECTANGLES OR NICE POLYGONS)

THM: LET $D \subset \mathbb{C}$ BE AN OPEN DISK IN $\mathbb{C}$ WITH $f: D \to \mathbb{C}$ CONTINUOUS. THEN IF FOR EVERY CLOSED TRIANGLE $T \subset D$, $\int_{\partial T} f\,dz = 0$, WE CONCLUDE f HAS A PRIMITIVE IN D.

LET $D = D_r(p)$, IE p IS THE CENTER.

DEFINE $F(z) = \int_{[p,z]} f(\zeta)\,d\zeta$



LET $z_0 \neq z$ AND LET T BE $\Delta(p, z, z_0)$

THEN

$$F(z) - F(z_0) = \int_{[p,z]} f(\zeta)\,d\zeta - \int_{[p,z_0]} f(\zeta)\,d\zeta = \int_{[z_0,z]} f(\zeta)\,d\zeta$$

AS BEFORE, WRITE

$$\frac{F(z) - F(z_0)}{z - z_0} - f(z_0), \quad \text{LET } z \to z_0$$

AND CONCLUDE $F'(z_0) = f(z_0)$ ∎

IF WE KNEW f' WERE CONTINUOUS, WE COULD USE GREEN'S THM TO ESTABLISH THIS TRIANGLE CONDITION. (f' IS CONTINUOUS FOR $f \in \mathcal{O}(U)$ BUT WE HAVEN'T PROVED IT YET)

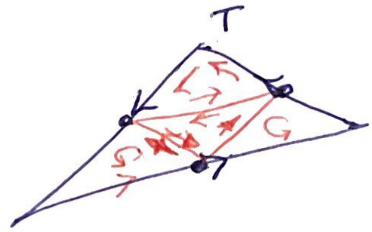
BUT WE HAVE

GOURSAT'S THM (1900) IF $f \in \mathcal{O}(U)$, THEN

$$\boxed{\int_{\partial T} f(z)\,dz = 0 \quad \text{FOR EVERY CLOSED TRIANGLE } T \subset U}$$

(USUALLY STATED WITH RECTANGLES. GOURSAT'S ORIGINAL WAS MORE COMPLICATED.)

PF/ FIX A CLOSED TRIANGLE $T \subset U$, AND LET $I = \int_{\partial T} f(z)\,dz$.



CONNECT THE MIDPOINTS OF EACH OF THE EDGES TO OBTAIN FOUR CONGRUENT TRIANGLES, EACH WITH $\frac{1}{2}$ DIAM(T).

NOW I IS THE SUM OF $\int_\gamma f(z)dz$ ALONG THE BOUNDARIES OF THESE FOUR, AND SO THERE IS A T_1 SO THAT $\left| \int_{\partial T_1} f(z)dz \right| \geq \frac{1}{4}|I|$.

CONTINUE INDUCTIVELY TO GET $T \supset T_1 \supset T_2 \supset T_3 \supset \cdots$ WITH $\text{DIAM}(T_n) = 2^{-n}\,\text{DIAM}(T)$ AND $\left| \int_{\partial T_n} f(z)dz \right| \geq \frac{1}{4^n}|I|$.

NOW, $\bigcap_{n=1}^{\infty} T_n = \{p\}$ WITH $p \in U$.

SINCE $f \in \mathcal{O}(U)$, $f'(p)$ EXISTS,

SO FOR ANY $\varepsilon > 0$, THERE IS A δ SO THAT

$$|z-p| < \delta \Rightarrow |f(z) - f(p) - f'(p)(z-p)| < \varepsilon |z-p|.$$

SINCE diam$(T_n) \to 0$, WE CAN CHOOSE n SO diam$(T_n) < \delta$,

AND SO FOR ANY $z \in \partial T_n$, $|z-p| < \text{diam}(T_n) < \delta$.

IN PARTICULAR $|f(z) - f(p) - f'(p)(z-p)| < \varepsilon \, \text{diam}(T_n)$.

NOTE THAT $f(p)z + \frac{1}{2} f'(p)(z-p)^2$ IS A PRIMITIVE FOR $f(p) - f'(p)(z-p)$ ON T_n, SO

$$\int_{\partial T_n} f(p) - f'(p)(z-p) = 0.$$

APPLYING THE ML INEQUALITY,

$$4^{-n}|I| \leq \left| \int_{\partial T_n} f(z)\,dz \right| = \left| \int_{\partial T_n} f(z) - \overbrace{f(p) - f'(p)(z-p)}^{0} \, dz \right|$$

$$\leq \varepsilon \, \text{diam}(T_n) \cdot \text{LENGTH}(\partial T_n)$$

$$= \varepsilon \frac{\text{diam}\, T}{2^n} \cdot \frac{\text{length}(\partial T)}{2^n}$$

SO $|I| \leq \varepsilon \, \text{diam}(T) \, \text{length}(T)$

SO $|I| = 0$

$\boxtimes$