FROM LAST TIME:

RECALL IF THE COMPLEX DERIVATIVE OF f AT $p \in \mathbb{C}$ IS $f'(p) = \alpha + i\beta$, THEN THE REAL DERIVATIVE OF f (VIEWED AS $f: \mathbb{R}^2 \to \mathbb{R}^2$) IS THE LINEAR MAP $Df_p = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$

MORE FORMALLY:

THEOREM LET $f = u + iv$ BE DEFINED IN A NEIGHBORHOOD OF $p \in \mathbb{C}$. THEN THE FOLLOWING ARE EQUIVALENT:

(i) THE COMPLEX DERIVATIVE $f'(p)$ EXISTS.

(ii) THE REAL DERIVATIVE Df_p EXISTS AND $f_{\bar{z}}(p) = 0$.

(iii) THE REAL DERIVATIVE Df_p EXISTS AND

$$u_x(p) = v_y(p) \qquad u_y(p) = -v_x(p) \qquad \text{(CAUCHY-RIEMANN EQ'NS)}$$

IN ANY OF THESE CASES,

$$f'(p) = f_z(p) = f_x(p) = -i f_y(p)$$

IF $f'(p) \neq 0$, $Df_p: \mathbb{R}^2 \to \mathbb{R}^2$ IS AN ORIENTATION PRESERVING CONFORMAL LINEAR TRANSFORMATION CONSISTING OF A ROTATION BY $\mathrm{Arg}(f'(p))$ FOLLOWED BY A SCALING BY $|f'(p)|$.

DEF LET $U \subset \mathbb{C}$ BE A NON-EMPTY OPEN SET.

$f : U \to \mathbb{C}$ IS HOLOMORPHIC IF $f'(p)$ EXISTS FOR ALL $z \in U$.

THE SET OF ALL HOLOMORPIC FUNCTIONS ON U IS DENOTED $\mathcal{O}(U)$.

IF $f \in \mathcal{O}(\mathbb{C})$, THEN f IS CALLED ENTIRE

EXAMPLES:

- ANY POLYNOMIAL IS AN ENTIRE FUNCTION.
- THE EXPONENTIAL $\exp : \mathbb{C} \to \mathbb{C}$ IS ENTIRE WITH
$$\exp(z) = e^{z} = e^{x} e^{iy} = e^{x}(\cos y + i \sin y)$$
- A RATIONAL FUNCTION $R(z) = \frac{p(z)}{q(z)}$ WITH $p(z)$ AND $q(z)$ POLYNOMIALS IS HOLOMORPHIC WHEREVER $q(z) \neq 0$.

  IN PARTICULAR, IF $r_1, r_2, \dots, r_n$ ARE THE ROOTS OF q, THEN
$$R(z) \in \mathcal{O}(\mathbb{C} \setminus \{r_1, r_2, \dots, r_n\})$$

REMARK: $f_{\bar{z}} = 0 \Rightarrow f \in \mathcal{O}(U)$ FOR ~~[illegible]~~ SOME $U \ni p$.

THIS IS WEAKER THAN REAL DIFFERENTIABILITY AT p.

FOR EXAMPLE, IF $f : U \to \mathbb{C}$ IS CONTINUOUS AND $f_z, f_{\bar{z}}$ EXIST EXCEPT FOR A COUNTABLE SUBSET OF U WITH $f_{\bar{z}} = 0$ a.e. ON U, THEN $f \in \mathcal{O}(U)$.

NOT FOR CLASS, JUST NOTES]

3

WHY CAN WE COMPUTE f_z AND $F_{\bar{z}}$ AS PARTIALS

EVEN THOUGH z & $\bar{z}$ ARE NOT INDEPENDENT?

LET $F : \mathbb{C}^2 \to \mathbb{C}$ BE GIVEN BY $F(z,w) = (f(z), \overline{f(\bar{w})})$

BE HOLOMORPIC IN EACH VARIABLE NEAR A POINT $(p, \bar{p})$.

THUS, THERE IS AN r SO THAT $z \mapsto F(z,w)$ IS HOLOMORPHIC ON $\mathbb{D}_r(p)$ FOR FIXED w, AND $w \mapsto F(z,w)$ IS HOLOMORPHIC ON $\mathbb{D}_r(\bar{p})$ FOR FIXED z.

WRITE $z = x+iy$ AND $w = s+it$.

THEN $F : (x,y,s,t)$ IS REAL DIFFERENTIABLE AS A FUNCTION $\mathbb{R}^4 \to \mathbb{R}^2$.

NOW CONSIDER $f(z) = F(z,\bar{z}) = F(x,y,x,-y)$

DEFINED IN A NEIGHBORHOOD OF p.

LET D_j BE PARTIAL DIFFERENTIATION BY THE j^{TH} VARIABLE, AND WE HAVE

$$f_x = D_1F + D_3F \qquad f_y = D_2F - D_4F$$

(EVALUATING f_* AT z AND D_*F AT $(z,\bar{z}) = (x,y,x,-y)$.)

FINALLY, $f_z = \frac{1}{2}(f_x - if_y)$ AND $f_{\bar{z}} = \frac{1}{2}(f_x + if_y)$ AS USUAL.

LETTING D_zF AND D_wF BE THE COMPLEX DERIVATIVE WRT EACH VARIABLE KEEPING THE OTHER FIXED, WE GET

$D_zF = (D_1F - iD_2F)/2$ AND $D_3F + iD_4F = 0$ (SINCE w FIXED)

$D_wF = (D_3F - iD_4F)/2$ AND $D_1F + iD_2F = 0$ (SINCE z FIXED)

THUS $f_z = D_zF$ AND $f_{\bar{z}} = D_wF$, to

4

FOR ANY $p \in \mathbb{C}$ AND SEQUENCE $\{a_n\}_{n=0}^{\infty}$ $a_n \in \mathbb{C}$,

WE HAVE A **POWER SERIES** $\sum_{n=0}^{\infty} a_n (z-p)^n$ IN z.

SUCH A SERIES WILL CONVERGE ON ~~THE~~ A DISK $\mathbb{D}_r(p)$ AND DIVERGE FOR $|z-p| > r$. (REQUIRES PROOF) FOR SOME r.

WE CAN COMPUTE r IF WE KNOW THE ASYMPTOTIC BEHAVIOR OF a_n AS $n \to \infty$.

<u>THM</u> (CAUCHY 1821) ~~AS POWER~~ FOR A SERIES $\sum_{n=0}^{\infty} a_n (z-p)^n$,

DEFINE $R = \dfrac{1}{\limsup \sqrt[n]{|a_n|}}$ (R MAY BE 0 OR $+\infty$)

THEN THE SERIES CONVERGES ABSOLUTELY AND UNIFORMLY IN THE DISK $\mathbb{D}_r(p)$ FOR ALL $0 \le r < R$ AND DIVERGES FOR ~~$r > R$~~. ANY z WITH $|z-p| > R$

R IS THE RADIUS OF CONVERGENCE OF THE SERIES $\sum_{n=0}^{\infty} a_n (z-p)^n$

PROOF ~~CONSIDE~~ LET $0 < r < s < R$.

BY THE DEFINITION OF R, THERE IS AN $N \geq 1$ SO THAT $|a_n s^n| < 1$ FOR $n \geq N$.

NOW LET $|z-p| < r$ AND WE HAVE

$$\sum_{n=N}^{\infty} |a_n| |z-p|^n \leq \sum_{n=N}^{\infty} |a_n| r^n = \sum_{n=N}^{\infty} |a_n| s^n \left(\frac{r}{s}\right)^n \leq \sum_{n=N}^{\infty} \left(\frac{r}{s}\right)^n$$

SINCE $\sum_{n=N}^{\infty} \left(\frac{r}{s}\right)^n$ IS A GEOMETRIC SERIES WITH RATIO < 1, IT CONVERGES, SO WE HAVE THE FIRST PART.

NOW SUPPOZE $|z-p| > R$. (ASSUMING $R < \infty$)

~~SINCE lim sup~~ LET $r > s > R$

THEN WE HAVE A SEQUENCE $|a_{n_j}| s^{n_j} > 1$ FOR ALL n_j

IF $|z-p| = r > s \geq R$ WE HAVE

$$|a_{n_j}| |z-p|^{n_j} > |a_{n_j}| s^{n_j} \left(\frac{r}{s}\right)^{n_j} > \left(\frac{r}{s}\right)^{n_j}.$$

BUT $\frac{r}{s} > 1$, SO THE SERIES DIVERGES.

WHAT HAPPENS ON THE BOUNDARY OF $\mathbb{D}_r(p)$ IS QUITE VARIABLE...

$\boxed{\begin{array}{l}\textbf{DEF}\quad \text{LET } U \subset \mathbb{C} \text{ BE NONEMPTY \& OPEN. } f : U \to \mathbb{C} \\ \text{IS } \underline{\text{COMPLEX ANALYTIC}} \text{ IF, FOR EVERY DISK} \\ \mathbb{D}_r(p) \subset U, \text{ THERE IS A CONVERGENT POWER SERIES} \\ \sum a_n (z-p)^n \text{ CONVERGING TO } f(z) \text{ WHEN } z \in \mathbb{D}_r(p).\end{array}}$

THM.

$\boxed{f \text{ IS ANALYTIC IN } U \iff f \in \mathcal{O}(U).}$ 6

FIRST HALF:

~~PA~~ ~~PA~~

THM LET $f: U \to \mathbb{C}$ BE COMPLEX ANALYTIC ON U. THEN

(i) $f \in \mathcal{O}(U)$, AND f' IS COMPLEX ANALYTIC IN U.

(ii) ALL HIGHER DERIVATIVES $f^{(k)}$, $k \geq 1$ EXIST AND ARE ANALYTIC IN U. THE POWER SERIES FOR $f^{(k)}$ CAN BE OBTAINED BY TERM-BY-TERM DIFFERENTIATION OF f.

(iii) THE COEFFICIENTS $\{a_n\}$ OF THE POWER SERIES FOR f ARE GIVEN BY $a_n = \dfrac{f^{(n)}(p)}{n!}$ $(n \geq 0)$

SINCE EACH a_n IS UNIQUELY DETERMINED BY f, ANY POWER SERIES IN $z-p$ WHICH CONVERGES TO f IN SOME DISK WITHIN U AND CENTERED AT p COINCIDES WITH. $f(z) = \sum_{n=0}^{\infty} a_n (z-p)^n$

(a_n AS IN (iii))

PROOF: LET $g(z) = \sum_{n=1}^{\infty} n a_n (z-p)^{n-1}$.

NOTICE THAT ~~[illegible]~~ THE POWER SERIES ABOVE HAS THE SAME RADIUS OF CONVERGENCE AS $\sum a_n (z-p)^n$ SINCE $\lim \sqrt[n]{n} = 1$,

SO g ALSO CONVERGES IN $\mathbb{D}_r(p)$.

IF WE SHOW THAT FOR ALL $z_0 \in \mathbb{D}_r(p)$, $f'(z_0)$ EXISTS AND

AND $f'(z_0) = g'(z_0)$, THEN WE ARE DONE,

SINCE $(i) \underset{\text{BY INDUCTION}}{\Rightarrow} (ii) \Rightarrow (iii)$.

BUT (i) IS A STRAIGHTFORWARD CALCULATION.

REPLACE $z-p$ BY z (IE TAKE $p=0$) AND FIX $z_0 \in \mathbb{D}_r$

CHOOSE $\varepsilon > 0$. THEN THERE IS AN s WITH

$$|z_0| < s < r \quad \text{SO THAT} \quad \sum_{n=N+1}^{\infty} n|a_n|^{n-1} s^{n-1} < \varepsilon$$

SINCE THE SERIES FOR g CONVERGES ABS. IN $\mathbb{D}_r$.

WE WANT TO SHOW $\dfrac{f(z) - f(z_0)}{z - z_0} - g(z_0)$ IS ARB. SMALL

BUT

$$= \sum_{n=2}^{\infty} a_n \left(\frac{z^n - z_0^n}{z - z_0} - n z_0^{n-1} \right)$$

LONG DIVISION

$$= \sum_{n=2}^{\infty} a_n \Big(\underbrace{z^{n-1} + z^{n-2} z_0 + z^{n-3} z_0^2 + \cdots + z_0^{n-1}}_{n \text{ TERMS}} - n z_0^{n-1} \Big)$$

THE SUM UP TO N TENDS TO 0 AS $z \to z_0$.

THE ABS. VALUE OF THE TAIL FROM $N+1$ IS BOUNDED BY $2n|a_n| s^{n-1}$ FOR $|z|<s$

BUT THIS IS LESS THAN 2ε.

THUS $f'(z_0) = g(z_0)$.

<u>EXAMPLE:</u> $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ CONVERGES ABS. FOR ALL z (SINCE $\lim \frac{1}{\sqrt[n]{n!}} \to 0$)

SO THIS IS AN ENTIRE FUNCTION WITH ~~VALUE~~ $f(0) = 1$ AND $f'(z) = f(z)$.

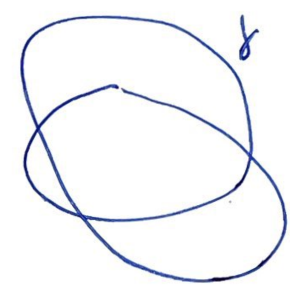
BUT ALSO $\exp(z) = e^{x}(\cos y + i \sin y)$ IS ENTIRE WITH $\exp(0) = 1$, AND $\exp' = \exp$.

IT IS EASY TO SHOW THAT $g(z) = \frac{f(z)}{\exp(z)}$ IS ALSO ENTIRE ($\exp \neq 0$ EVERYWHERE), AND BY THE QUOTIENT RULE $g(0) = 1$ WITH $g'(z) = 0$ FOR ALL z.

THUS $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

INTEGRATION

NOW WE TURN TO INTEGRATION ALONG AN ORIENTED CURVE γ. WE ASSUME γ IS PIECEWISE SMOOTH (MERELY RECTIFIABLE IS ENOUGH, BUT MOSTLY NOT WORTH THE TROUBLE)



MOST RESULTS ONLY DEPEND ON THE HOMOLOGY CLASS OF γ — IN FACT, ONE CAN DEVELOP A HOMOLOGY THEORY (FOR COMPLEX SURFACES AT LEAST) FROM INTEGRALS ALONG CURVES.

~~DEF~~: AS USUAL $U \subset \mathbb{C}$ OPEN, AND A CURVE $\gamma \subset U$ IS A CONTINUOUS $\gamma : [a,b] \to U$, IT IS CLOSED IF $\gamma(a) = \gamma(b)$.

(BOOK SOMETIMES WRITE $|\gamma|$ FOR THE IMAGE OF γ, IE
$\text{Image}\,\gamma = |\gamma| = \{ z \mid z = \gamma(t) \text{ FOR SOME } t \in [a,b] \}$)

γ IS PIECEWISE C^1 IF THERE ARE $a = t_0 < t_1 < t_2 \cdots < t_n = b$ SO THAT
- γ IS CONTINUOUSLY DIFFERENTIABLE ON (t_{k-1}, t_k)
- $\lim_{t \to t_{k-1}^+} \gamma'(t)$ AND $\lim_{t \to t_k^-} \gamma'(t)$ EXIST.

DEF LET $\gamma : [a,b] \to \mathbb{C}$ BE A PIECEWISE SMOOTH CURVE WITH $f : \text{Image}\,|\gamma| \to \mathbb{C}$ CONTINUOUS. THEN THE INTEGRAL ALONG γ OF f IS

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

NOTE γ' IS DEFINED FOR ALL BUT FINITELY MANY POINTS OF $[a,b]$

BY BREAKING f, γ UP INTO $f = u + iv$

AND $\gamma = x + iy$

AND TAKING REAL AND IMAGINARY PARTS, WE CAN CAST $\int_\gamma f(z)\,dz$ AS A PATH INTEGRAL FROM CALCULUS,

$$\boxed{\int_\gamma f(z)\,dz = \int_\gamma (u\,dx - v\,dy) + \int_\gamma v\,dx + u\,dy}$$

WE CAN ALWAYS ASSUME $[a,b] = [0,1]$.

- WE CAN "RUN γ BACKWARDS".

  GIVEN $\gamma : [0,1] \to \mathbb{C}$, $\gamma^{-} : [0,1] \to \mathbb{C}$

  SATISFIES $\gamma^{-}(t) = \gamma(1-t)$

  AND $$\int_{\gamma^-} f(z)\,dz = -\int_{\gamma} f(z)\,dz$$

- CONCATENATING TWO CURVES ~~IS~~ THE "PRODUCT" OF γ, η.

  IE IF $\gamma(1) = \eta(0)$, THEN $\gamma \cdot \eta : [0,1] \to \mathbb{C}$

  IS JUST
  $$(\gamma \cdot \eta)(t) = \begin{cases} \gamma(2t) & t \in [0, 1/2] \\ \eta(2t-1) & t \in [1/2, 1]. \end{cases}$$

  AND $$\boxed{\int_{\gamma \cdot \eta} f(z)\,dz = \int_{\gamma} f(z)\,dz + \int_{\eta} f(z)\,dz}$$

  AS LONG AS $f(\gamma \cdot \eta)$ IS CONTINUOUS

- CLOSELY RELATED IS THE <u>PATH INTEGRAL</u> $f : U \to \mathbb{C}$

  AS IN CALCULUS
  $$\int_{\gamma} f(z)\,|dz| = \int_0^1 f(\gamma(t))\,|\gamma'(t)|\,dt$$

  AND TAKING $f \equiv 1$ GIVES THE <u>LENGTH</u> OF γ

  $$|\gamma| = \text{length}(\gamma) = \int_{\gamma} |dz| = \int_0^1 |\gamma'(t)|\,dt$$

SINCE $\left|\int_\gamma f(z)\,dz\right| \le \int_\gamma |f(z)|\,|dz|$,

WE GET THE VERY USEFUL INEQUALITY

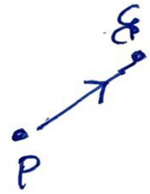
$$\left|\int_\gamma f(z)\,dz\right| \le \sup_{z\in\gamma}|f(z)|\cdot \text{length}(\gamma)$$

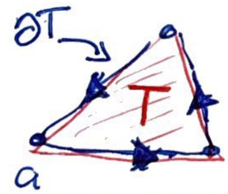
SOMETIMES CALLED THE <u>ML-INEQUALITY</u>

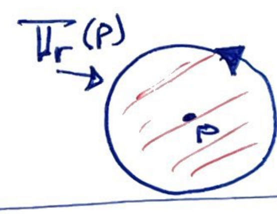
(MAXIMUM · LENGTH)

IT IS PARTICULARLY USEFUL TO INTEGRATE ALONG

SEGMENTS $[p,q]$ BOUNDARIES OF TRIANGLES (a,b,c) CIRCLES $\mathbb{T}_r(p) = \partial\mathbb{D}_r(p)$







UNLESS OTHERWISE STATED, THE CLOSED CURVE IS ORIENTED COUNTER-CLOCKWISE