COMPLEX ANALYSIS, MON, 1/23/23 (1)

CLASS MEETS MW 11:30 – 12:50
OK? CHECK [FORMALLY, 11:45] OR 1

BASIC ADMIN STUFF:

- GRADING:
  - (WEEKLY) HOMEWORK 30%
  - MIDTERM (MID MARCH) 30%
  - FINAL FRI 5/12, 11:15am 40%

- TEXT ZAKERI: A COURSE IN COMPLEX ANALYSIS
  PRINCETON, 2021

  OTHER REFS: AHLFORS, CONWAY, ETC.

GRADER: WILL LIM (MATH 5-125B)
ME! SCOTT SUTHERLAND (MATH 5-112)

BACKGROUND: ANALYSIS, TOPOLOGY. PREV. COMPLEX HELPFUL.

COVERAGE: SEE GRAD CORE COURSE/WEB.
BUT EFFECTIVELY, MOST OF CH 1–7 OF ZAKERI,
MORE IF WE CAN.

BASICS: 1. HOLOMORPHIC FUNCS, DEF OF INTEGRATION, ETC
2. TOPOLOGICAL ASPECTS OF CAUCHY: HOMOTOPY, ~~winding~~
COVERING BY EXP, WINDING #
3. MEROMORPHIC FUNCS: SINGULARITIES, RIEMANN SPHERE
LAURENT SERIES & RESIDUES, ARGUMENT PRINCIPLE
4. MÖBIUS MAPS, SCHWARZ LEMMA
5. NORMALITY & CONVERGENCE
6. CONFORMAL MAPPINGS: RIEMANN MAPPING
THM, SCHLICT FUNCTIONS
7. HARMONIC FUNCTIONS

MORE IF TIME.

MOST BASIC:

WE EXTEND ~~R~~ $\mathbb{R}$ BY ADDING i WITH $i^2 = -1$

- i IS THE "IMAGINARY UNIT". NAMED IMAGINARY BY DESCARTES (1596-1650) IN 1637 SINCE NO REAL # SATISFIED THIS RELATION
- CARDANO (1501-1576) (GENERALLY VIEWED AS RESPONSIBLE FOR SYSTEMATIC USE OF NEGATIVE #S)

  USED THEM TO SOLVE CUBICS, BUT CALLED USING THEM "MENTAL TORTURE" AND SAID THEY WERE "AS SUBTLE AS THEY WERE USELESS".

  BOTH TARTAGLIA AND CARDANO NEEDED THEM TO MAKE SENSE OF EQUATIONS FOR SOLVING CUBIC AND QUARTIC POLYNOMIALS, BUT NO SYSTEMATIC FORMALIZATION OF ALGEBRA.
- BOMBELLI (1526-1572) FORMALIZED THE ALGEBRA OF $\mathbb{C}$, (CALLING i "PLUS OF MINUS" AND $-i$ "MINUS OF MINUS")
- THE INTRODUCTION OF i FOR $\sqrt{-1}$ IN THE 17TH CENTURY SEEMS TO BE IN PART TO AVOID CONFUSION DUE TO

$$\sqrt{ab} = \sqrt{a}\cdot\sqrt{b} \quad \text{VS} \quad (\sqrt{-1})^2 = \sqrt{-1}\cdot\sqrt{-1} = -1 \overset{?}{\neq} \sqrt{(-1)\cdot(-1)}$$

- EULER (1707-1783) SEEMS RESPONSIBLE FOR THE NOTATION i, (AS WELL AS π AND THE INTRODUCTION OF e) AND EULER'S FORMULA $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$ EXTENDING DE MOIVRE'S (1667-1754) $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$

AS WELL AS TONS OF OTHER MATH

HISTORY, CONT'D

ANTICIPATED BY WALLIS (1616-1703)

- NOTION OF A COMPLEX NUMBER AS A POINT IN THE PLANE ~~BE~~ DESCRIBED BY WESSEL (1799) BUT BETTER KNOWN BY ARGAND (1806), INCLUDING A PROOF OF THE FUND. THM OF ALGEBRA.

  GAUSS (1777-1855) HAD PROVIDED AN ESSENTIALLY TOPOLOGICAL PROOF IN 1797, BUT HAD DOUBTS ABOUT THE USE OF $\sqrt{-1}$. IN 1831 HE PUBLISHED A TREATISE ON THE GEOMETRIC INTERPRETATION OF COMPLEX #s, INTRODUCING THE TERM "COMPLEX" (AND COMPLAINED ABOUT THE "DARKNESS AND MYSTERY" DUE TO THE TERM "IMAGINARY" FOR i, SAYING INSTEAD 1 = DIRECT UNITY, -1 = INVERSE UNITY, i = LATERAL UNITY)

- HAMILTON (1805-1865) IN 1831 GAVE AN ALGEBRAIC FORMULATION USING ORDERED PAIRS.

- THE 19TH CENTURY SAW THE FLOWERING OF COMPLEX ~~ANALY~~ FUNCTION THEORY.
  - CAUCHY (1789-1857) GENERALLY ATTRIBUTED CREATION IN AN 1814 MEMOIR (PUBL 1825)
    - DISCUSSES ANALYTIC FUNCTIONS [NOT WITH THAT NAME] & CONTOUR INTEGRALS (POISSON USED IN 1820)
  - RIEMANN (1826-1866) ~~LAID~~ ESTABLISHED GEOMETRIC FOUNDATION IN HIS DISSERTATION (RIEMANN SURFACES) RIEMANN MAPPING THM, ETC.
  - TONS MORE ...

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OK, LETS START FOR REAL NOW.

① NOTATION

- $\mathbb{C} = \{x+iy \mid x, y \in \mathbb{R}\}$ WITH i^2 [FIXED] $= -1$

- IF $z \in \mathbb{C}$, $\boxed{\begin{array}{l} x = \mathrm{Re}(z) \\ y = \mathrm{Im}(z) \\ \bar{z} = x - iy \end{array}}$ "THE REAL PART OF Z" "THE IMAGINARY PART OF Z" "THE COMPLEX CONJUGATE OF Z"

NOTE $\mathrm{Re}(z) = \frac{1}{2}(z+\bar{z})$, $\mathrm{Im}(z) = \frac{1}{2}(z-\bar{z})$

$z = \bar{z} \iff z \in \mathbb{R}$.

- THE ABSOLUTE VALUE OF Z (AKA MODULUS, NORM)

$$\boxed{|z| = \sqrt{x^2+y^2}} \qquad (\text{SO } z\bar{z} = |z|^2 \in \mathbb{R})$$

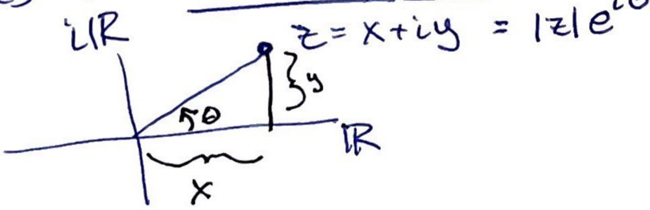
- IF $z \neq 0$, $\frac{z}{|z|}$ HAS NORM 1, SO $\exists \theta \in \mathbb{R}$

WITH $z = \cos\theta + i\sin\theta = e^{i\theta}$

θ IS AN ARGUMENT OF Z, UNIQUE UP TO INTEGER MULTIPLE OF 2π

$$\boxed{\theta = \arg(z)}$$

- THIS GIVES THE POLAR FORM OF Z



$\mathbb{D}$ IS THE OPEN UNIT DISK $= \{ z \in \mathbb{C} \mid |z| < 1 \}$

$\mathbb{D}(p,r) = \mathbb{D}_r(p) = \{ z \in \mathbb{C} \mid |z-p| < r \}$

UNLESS OTHERWISE STATED:

$z = x + iy$ MEANS $x, y \in \mathbb{R}$.

FUNCTION $f = u + iv$ ASSUMES u, v REAL VALUED

WRITE $u = \mathrm{Re}(f)$, $v = \mathrm{Im}(f)$.

DIFFERENTIABILITY

DEF: SPOZE f IS COMPLEX VALUED & DEFINED IN AN OPEN NEIGHBORHOOD OF $p \in \mathbb{C}$. THEN f IS (COMPLEX) DIFFERENTIABLE IF

$$\lim_{z \to p} \frac{f(z) - f(p)}{z - p} \quad (\triangleq f'(p)) \quad \text{EXISTS.}$$

$f'(p)$ IS THE (COMPLEX) DERIVATIVE OF f AT p

EASY FACT: f DIFFERENTIABLE AT $p \Rightarrow f$ CONTINUOUS AT p

SINCE

$$\lim_{z \to p} \left(f(z) - f(p) \right) = \lim_{z \to p} \frac{f(z) - f(p)}{z - p} (z - p) = f'(p) (0) = 0$$

$\boxtimes$

THE "USUAL" RULES FOR DIFFERENTIATION FROM CALCULUS STILL HOLD FOR COMPLEX FUNCTIONS.

ie

THM

(i) LET f AND g BE DIFF. AT p. THEN

$$(f+g)'(p) = f'(p) + g'(p)$$

$$(fg)'(p) = f'(p)g(p) + f(p)g'(p)$$

IF ALSO $g'(p) \neq 0$,

$$\left(\frac{f}{g}\right)'(p) = \frac{f'(p)g(p) - f(p)g'(p)}{[g(p)]^2}$$

(ii) SPOZE g DIFF AT p, f DIFF AT $g(p)$. THEN

$$(f \circ g)'(p) = f'(g(p))\, g'(p)$$

PROOF IS ESSENTIALLY THE SAME AS FOR REAL VALUED FUNCTIONS

COR EVERY POLYNOMIAL $f(z) = \sum_{n=0}^{d} a_n z^n$ $(a_n \in \mathbb{C})$ IS DIFFERENTIABLE AT ALL $z \in \mathbb{C}$, WITH

$$f'(z) = \sum_{n=1}^{d} n a_n z^{n-1}$$

PROOF THE IDENTITY FUNCTION $f(z) = z$ SATISFIES $f'(z) = 1$ (FROM DEFINITION) FOR ALL $z \in \mathbb{C}$. SIMILARLY, THE DERIVATIVE OF A CONSTANT FUNCTION IS 0 FROM THE DEF.

REPEATEDLY APPLY THE THM TO GET THE RESULT.

EXAMPLE $f(z) = z\bar{z} = |z|^2$ IS DIFF ONLY AT $z=0$.

IF $z \neq 0$, $\dfrac{f(p+z) - f(p)}{z} = \dfrac{(p+z)(\overline{p+z}) - p\bar{p}}{z} = p\dfrac{\bar{z}}{z} + \bar{p} + \bar{z}$

BUT $\dfrac{\bar{z}}{z}$ DOES NOT HAVE A LIMIT AS $z \to 0$.

FOR EXAMPLE, IF $z \in \mathbb{R}$, $\lim\limits_{z\to 0} \dfrac{\bar{z}}{z} = 1$

IF $z \in i\mathbb{R}$ $\lim\limits_{z\to 0} \dfrac{\bar{z}}{z} = -1$

THUS $f'(p)$ EXISTS ONLY IF $p = 0$. ☒

OK, SO WHAT? LET $f: \mathbb{C} \to \mathbb{C}$, BUT VIEW $f: \mathbb{R}^2 \to \mathbb{R}^2$ AND COMPARE $f'(z)$ WITH $Df(x,y)$.

SPECIFICALLY WRITE

$$f(x,y) = u(x,y) + i v(x,y)$$

THEN

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \qquad \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$$

INTRODUCE

$$\boxed{\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \qquad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)}$$

SO

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}} \qquad \frac{\partial f}{\partial y} = i\left(\frac{\partial f}{\partial z} - \frac{\partial f}{\partial \bar{z}}\right)$$

$\frac{\partial f}{\partial z}$ AND $\frac{\partial f}{\partial \bar{z}}$ ARE NOT REALLY PARTIALS, SINCE $z, \bar{z}$ NOT INDEPENDENT.

BUT USEFUL NOTATION

AND CAN MAKE SENSE (LATER)

EXAMPLE WRITE $\log|z| = \frac{1}{2}\log(z\bar{z}) = \frac{1}{2}\log(x^2+y^2)$.

THEN
$$\frac{\partial}{\partial z}\log|z| = \frac{1}{4}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\log(x^2+y^2) = \frac{1}{2}\,\frac{x-iy}{x^2+y^2} = \frac{1}{2z}$$
$$\frac{\partial}{\partial \bar{z}}\log|z| = \frac{1}{4}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\log(x^2+y^2) = \frac{1}{2}\left(\frac{x+iy}{x^2+y^2}\right) = \frac{1}{2\bar{z}},$$

BUT ALSO, TAKING PARTIALS FORMALLY:
$$\frac{\partial}{\partial z}\left(\frac{1}{2}\log(z\bar{z})\right) = \frac{1}{2}\,\frac{\bar{z}}{z\bar{z}} = \frac{1}{2z}$$
$$\frac{\partial}{\partial \bar{z}}\left(\frac{1}{2}\log(z\bar{z})\right) = \frac{1}{2}\,\frac{z}{z\bar{z}} = \frac{1}{2\bar{z}}.$$
MAGIC! $\boxtimes$

NOW LET'S LOOK AT $Df_p : \mathbb{R}^2 \to \mathbb{R}^2$ AT $p \in \mathbb{R}^2$ THE REAL DERIVATIVE OF f. $Df_p(x,y)$ IS THE UNIQUE LINEAR MAP SO
$$\lim_{(x,y)\to(0,0)} \frac{\| f(p+(x,y)) - f(p) - Df_p(x,y) \|}{\|(x,y)\|} = 0$$

WHERE $\|(x,y)\| = \sqrt{x^2+y^2}$

OR, VIA TAYLOR POLY,
$$f(p+(x,y)) = f(p) + Df_p(x,y) + \varepsilon(x,y)$$
WHERE $\frac{\|\varepsilon(x,y)\|}{\|(x,y)\|} \to 0$ AS $(x,y)\to(0,0)$

IN THE STANDARD BASIS OF $\mathbb{R}^2$,

$$Df_p = \begin{pmatrix} \frac{\partial u}{\partial x}(p) & \frac{\partial u}{\partial y}(p) \\ \frac{\partial v}{\partial x}(p) & \frac{\partial v}{\partial x}(p) \end{pmatrix}$$

OR, WRITING $f_x = \frac{\partial f}{\partial x}$ ETC, ...

$$Df_p(x,y) = \begin{pmatrix} u_x(p) & u_y(p) \\ v_x(p) & v_y(p) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\,u_x(p) + y\,u_y(p) \\ x\,v_x(p) + y\,v_y(p) \end{pmatrix}$$

AS COMPLEX NUMBERS, WE HAVE

$$(x u_x(p) + y u_y(p)) + i(x v_x(p) + y v_y(p)) = x f_x(p) + y f_y(p)$$

$$=$$

$$= \underbrace{\tfrac{1}{2}(z+\bar{z})}_{x}\left(f_z(p) + f_{\bar{z}}(p)\right) + \tfrac{1}{2}(z-\bar{z})\left(f_z(p) - f_{\bar{z}}(p)\right)$$

$$= z f_z(p) + \bar{z} f_{\bar{z}}(p)$$

WRITING TAYLOR IN THIS NOTATION

$$\boxed{f(p+z) = f(p) + z f_z(p) + \bar{z} f_z(p) + \varepsilon(z)}$$

WHERE $\lim_{z\to 0} \frac{\varepsilon(z)}{z} = 0$

IF $f_{\bar{z}}(p) = 0$, WE HAVE

$$\frac{f(p+z) - f(p)}{z} = f_z(p) + \frac{\varepsilon(z)}{z}$$

AS $z \to 0$, $f'(p)$ EXISTS AND EQUALS $f_z(p)$.

<u>CONVERSELY</u> SPOZE $f'(P)$ EXISTS.

THEN

$$f(P+z) = f(P) + f'(P)z + \varepsilon(z) \qquad \left(\text{WITH } \frac{\varepsilon(z)}{z} \to 0 \text{ AS } z \to 0 \right)$$

SO $z \mapsto f'(z)z$, VIEWED AS A LINEAR MAP $\mathbb{R}^2 \to \mathbb{R}^2$ AGREES WITH THE REAL DERIVATIVE $Df_P(z)$

WRITING $f'(P) = \alpha + i\beta$,

$$Df_P = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

WHERE $\alpha = u_x(P) = v_y(P) \qquad \beta = -u_y(P) = v_x(P)$

AND $f_{\bar z}(P) = \frac{1}{2}\left(f_x(P) + i f_y(P)\right) = \frac{1}{2}\left(\alpha + i\beta + i(-\beta + i\alpha)\right)$

$$= \frac{1}{2}\left(\alpha - \alpha + i(\beta - \beta)\right) = 0.$$

THUS

<u>THEOREM</u> LET $f = u + iv$ BE DEFINED IN A NEIGHBORHOOD OF $P \in \mathbb{C}$, THEN THE FOLLOWING ARE EQUIVALENTS

(i) THE COMPLEX DERIVATIVE $f'(P)$ EXISTS

(ii) THE REAL DERIVATIVE Df_P EXISTS <u>AND</u> $f_{\bar z}(P) = 0$

(iii) THE REAL DERIVATIVE Df_P EXISTS AND

$$u_x(P) = v_y(P), \quad u_y(P) = -v_x(P)$$

IN ANY OF THESE CASES,

$$f'(P) = f_z(P) = f_x(P) = -i f_y(P)$$

LETS LOOK AT Df_p A BIT MORE:

ASSUME $f'(p) = \alpha + i\beta$, AND $f'(p) \neq 0$.

THEN $\det Df_p = \alpha^2 + \beta^2$, AND

$$Df_p = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} = \underbrace{\begin{pmatrix} \sqrt{\alpha^2+\beta^2} & 0 \\ 0 & \sqrt{\alpha^2+\beta^2} \end{pmatrix}}_{\text{DILATION BY } \sqrt{\alpha^2+\beta^2}} \underbrace{\begin{pmatrix} \frac{\alpha}{\sqrt{\alpha^2+\beta^2}} & \frac{-\beta}{\sqrt{\alpha^2+\beta^2}} \\ \frac{\beta}{\sqrt{\alpha^2+\beta^2}} & \frac{\alpha}{\sqrt{\alpha^2+\beta^2}} \end{pmatrix}}_{\text{ROTATION BY } \Theta = \arccos\left(\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}\right)}$$

THAT IS Df_p IS A ROTATION BY $\operatorname{Arg}(f'(p))$ FOLLOWED BY A SCALING BY $|f'(p)|$

THIS IS A <u>CONFORMAL MAP</u> IN THAT IT PRESERVES ANGLES (BUT NOT SCALES NECESSARILY)

<u>COR</u> SPOZE $f'(p)$ IS NONZERO. THEN THE REAL DERIVATIVE $Df_p : \mathbb{R}^2 \to \mathbb{R}^2$ IS AN ORIENTATION PRESERVING CONFORMAL LINEAR TRANSFORMATION

DEF

LET $U \subset \mathbb{C}$ BE NON-EMPTY AND OPEN.
$f: U \to \mathbb{C}$ IS <u>HOLOMORPHIC</u> IF $f'(p)$ EXISTS FOR ALL $z \in U$

THE SET OF ALL HOLOMORPHIC FUNCTIONS ON U IS DENOTED $\mathcal{O}(U)$.

IF $f \in \mathcal{O}(\mathbb{C})$, THEN f IS CALLED <u>ENTIRE</u>

EXAMPLES

- <u>POLYNOMIALS</u> ARE ENTIRE FUNCTIONS.
- THE <u>EXPONENTIAL</u> $\exp: \mathbb{C} \to \mathbb{C}$ IS ENTIRE, WITH

$$\exp(z) = e^z = e^x e^{iy} = e^x(\cos y + i \sin y)$$

- A <u>RATIONAL FUNCTION</u> $R(z) = \dfrac{P(z)}{g(z)}$ WITH $P(z), g(z)$ POLYNOMIAL IS HOLOMORPHIC WHERE $g(z) \neq 0$.

IN PARTICULAR, IF g IS NOT IDENTICALLY 0 AND $r_1, r_2, \dots, r_n$ ARE THE ROOTS OF g, THEN

$$R(z) \in \mathcal{O}(\mathbb{C} \setminus \{r_1, r_2, \dots, r_n\}).$$

REMARK $f_{\bar{z}} = 0 \Rightarrow f \in \mathcal{O}(U)$ IS WEAKER THAN REAL DIFFERENTIABILITY OF f.

FOR EXAMPLE, IF • $f: U \to \mathbb{C}$ IS CONTINUOUS, • f_z AND $f_{\bar{z}}$ EXIST FOR ALL BUT A COUNTABLY SUBSET OF U
AND $f_{\bar{z}} = 0$ FOR ALMOST EVERY $z \in U$, THEN $f \in \mathcal{O}(U)$