MAT 536 Solutions to Midterm

10 pts 1. Let $U \subset \mathbb{C}$ be a simply connected domain, and let f be meromorphic on U. Suppose that for each pole p of f, the residue of f at p is zero. Show that f has a primitive F on U.

Solution: By Morera's theorem, if a continuous function f on U satisfies $\int_\gamma f(z)\,dz = 0$ for every closed, piecewise smooth curve γ in U, it is holomorphic in U and hence has a primitive.

Take any curve γ so that no pole of f lies on $\{\gamma\}$, and let E be the set of poles of f lying within $\{\gamma\}$. Since f is meromorphic, each pole of f is an isolated singularity, and we can apply the residue theorem:

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{p\in E} W(\gamma,p)\,\mathrm{res}(f,p) \quad .$$

But since $\mathrm{res}(f,p) = 0$ for each p, this integral is zero.

If a pole p_0 lies on $\{\gamma\}$, observe that we can perturb γ slightly to obtain γ^+ and γ^- so that $p_0 \notin \{\gamma^\pm\}$, with the poles within γ^+ being E, and those within γ^- being $E \setminus p_0$. The above argument applies to the integrals over γ^+ and γ^-, and hence the integral over γ is also zero.

Since we have shown $f \in \mathcal{O}(U)$, f has a primitive.

You could also have done this by showing that any pole with residue zero is removable, or by constructing the primitive directly (fix a basepoint $z_0 \in U$, and let the primitive $F(z)$ be the integral over a path from z_0 to z. It isn't hard to show that f is the derivative of F, just a bit long.

10 pts 2. Let $\qquad f(z) = z^9 + z^5 - 9z^3 + 3z + i/3 \quad .$

How many zeros does f have in the annulus $1/3 < |z| < 1$? Fully justify your answer.

Solution: We will use Rouché's theorem, which says that if $f, g \in \mathcal{O}(U)$ with U a simply-connected domain, and γ a Jordan curve in U such that $|f(z) - g(z)| < |g(z)|$ for $z \in \gamma$, then $|f|$ and $|g|$ have the same number of zeros (with multiplicity) in $\mathrm{int}(\gamma)$.

First, let's determine the number of zeros in $\mathbb{D}$. If $|z| = 1$, then take $g(z) = -9z^3$. Then we have

$$|f(z) - g(z)| = |z^9 + z^5 + 3z + i/3| \le 1 + 1 + 3 + 1/3 < 9 = |-9z^3|\,,$$

so f has three zeros in $\mathbb{D}$.

Now on $\mathbb{D}_{1/3}$ we can take $g(z) = 3z$ and observe that

$$|f(z) - g(z)| = |z^9 + z^5 - 9z^3 + i/3| \le \frac{1}{3^9} + \frac{1}{3^5} + \frac{1}{3} + \frac{1}{3} < 1 = |3z|\,,$$

giving just one zero with modulus less than $1/3$.

This means there are $3 - 1 = 2$ zeros of f in the annulus $\{\,1/3 < |\,z| < 1\,\}$.

In case you actually care, the roots of f (sorted by modulus) are approximately

$$\{\,-0.107i,\ \pm 0.598 + 0.0569i,\ \pm 1.33 - 0.0564i,\ \pm 0.755 + 1.29i,\ \pm 0.7595 - 1.276i\,\} \quad .$$

10 pts 3. Let f be a non-constant entire function. Show the that $f(\mathbb{C})$ is dense in $\mathbb{C}$.

Solution: There are three possibilities for the behavior of f at infinity.

If $f(\infty) = a \in \mathbb{C}$, then by the continuity of f, $|f|$ is bounded and hence constant by Liouville's theorem. So this doesn't happen.

If $f(\infty) = \infty$, f has a pole at ∞. Since f is entire, it must be a polynomial, and so $f(\mathbb{C}) = \mathbb{C}$.

Otherwise, there must be an essential singularity at infinity. By the Casorati-Weirstrass Theorem, the image of any deleted neighborhood of an essential singularity is dense in $\mathbb{C}$. (In fact, by Picard's theorem, it misses at most one point in $\mathbb{C}$.)

Strictly speaking, you shouldn't use either Picard's theorem here, since a proof relies on material we haven't covered yet (such as normal families). I should have put "don't use Picard's theorem" in the problem, but I didn't, so I won't penalize you if you use it properly.

10 pts 4. Let f be meromorphic in a simply connected domain U, and let p be a pole of f. Show that no pole p of f is also a pole of $e^f = exp \circ f$.

Solution: Since f is meromorphic, any pole p is an isolated singularity, and hence in some disk $\mathbb{D}_r(p)$, we have $f(z) = g(z)/(z-p)^m$ for some $m \in \mathbb{Z}^+$ and with $g(p) \neq 0$. Since $e^z = \sum_{n=0}^{\infty} z^n/n!$, we have

$$e^{f(z)} = \sum_{n=0}^{\infty} \frac{(g(z))^n}{n!(z-p)^{nm}} \quad .$$

Since the series for e^f has infinitly many negative terms, it has an essential singularity at p, rather than a pole.