MAT536 Homework 12
Due Wednesday, May 3

1. Show that if f and g are holomorphic in a domain $U \subset \mathbb{C}$ and the product $f\overline{g} : U \to \mathbb{C}$ is harmonic, then either f or g must be constant in U.

2. Let U be a domain in $\mathbb{C}$ on which the complex-valued function f is harmonic. Prove that if $|f(z)|$ is constant on U, then f is constant on U.

3. Suppose $f : \mathbb{D} \to \mathbb{C}$ is harmonic, with $zf(z)$ also harmonic. Show that f is holomorphic on $\mathbb{D}$.

4. **Harnack's inequalities** state that if $u : \mathbb{D}_R(p) \to \mathbb{R}$ is a positive function, then if $|z-p| = r$ with $0 \leq r < R$, we have

$$\frac{R-r}{R+r}u(p) \leq u(z) \leq \frac{R+r}{R-r}u(p) \quad .$$

The proof follows from applying the Poisson integral formula in $\mathbb{D}_s(p)$ with $r < s < R$ and taking the limit as $s \to R$.

Suppose u is a positive harmonic function in the unit disk $\mathbb{D}$ with $u(0) = 1$. Use Harnack's inequalities to give upper and lower bounds on $u(1/2)$, and show that your bounds are optimal.

5. Let a and b be real positive constants, and suppose $u : \mathbb{C} \to \mathbb{R}$ is a harmonic function satisfying

$$u(z) \leq a\big|\log|z|\big| + b \quad .$$

Show u must be constant.