MAT536 Homework 11

Due Wednesday, April 26

1. Establish the **subordination principle**: Let $f : \mathbb{D} \to U$ be a Riemann map with $f(0) = 0$ and let $g : \mathbb{D} \to U$ be any holomorphic map with $g(0) = 0$. Show that $g(\mathbb{D}_r(0)) \subset f(\mathbb{D}_r(0))$ for all $0 < r < 1$.

2. Suppose $f : \mathbb{D} \to U$ is conformal and $0 \notin U$.

(a) Prove that $|f'(0)| \leq 4|f(0)|$.

(b) More generally, prove that $|f'(z)| \leq \dfrac{4|f(z)|}{1-|z|^2}$ for $z \in \mathbb{D}$.

3. Show that $f \in \mathcal{S}$ commutes with $z \mapsto -z$ if and only if it is a holomorphic square root of $g(z^2)$ for some $g \in \mathcal{S}$. (Recall that $\mathcal{S}$ is the family of schlict functions).

4. Let $f \in \mathcal{S}$, and let $A(r)$ denote the area of $f(\mathbb{D}_r(0))$ for $0 < r < 1$. Show that $A(r) \geq \pi r^2$.
Hint: use the power series for f in $A(r) = \iint_{\mathbb{D}_r} |f'(z)|^2 dx\,dy = \int_0^r \int_0^{2\pi} f'(\rho e^{it})\overline{f'(\rho e^{it})}\,\rho\, d\rho\, dt$ and estimate from below.

5. Although a Riemann map may fail to have radial limits at some points of the unit circle, its inverse behaves nicer in the following sense. If $g : U \to \mathbb{D}$ is conformal and the curve $\xi : [0,1) \to U$ lands at a point $q \in \partial\, U$, then the image $g \circ \xi : [0,1) \to \mathbb{D}$ lands at a well-defined point of $\partial\, \mathbb{D}$. Moreover, two curves in U with distinct landing points map to curves in $\mathbb{D}$ with distinct landing points.
Hint: Assume p_1 and p_2 are distinct accumulation points of $g(\xi(t))$ as $t \to 1$, construct cross-cuts γ_n and η_n near p_1 and p_2 and show that their images must lie in an arbitrarily small neighborhood of q for large n, arriving at a contradiction.