MAT536 Homework 10

Due Wednesday, April 19

1. Let $P(z)$ be a polynomial of degree at least 2, with $P(0)=0$ and $P'(0)=\alpha$. Let $P^{\circ n}$ denote the n-fold composition of P with itself, that is, $P^{\circ n}(z)=(P\circ P\circ\cdots\circ P)(z)$.
Show that the family of iterates

$$\mathcal{P}=\{\,P^{\circ n}\mid n\in\mathbb{Z}^{+}\,\}$$

is normal in some neighborhood of 0 if $|\alpha|<1$. Also show that $\mathcal{P}$ cannot be normal in any neighborhood of 0 if $|\alpha|>1$ or $\alpha^k=1$ for some positive integer k.

2. Let f_n be meromorphic on a domain U with $f_n\to\infty$ in $\mathcal{C}(U,\widehat{\mathbb{C}})$. As the examples $z\mapsto z+n$ on $U=\mathbb{C}$ and $z\mapsto nz$ on $U=\mathbb{C}^*$ show, not much can be said about f_n' in general. Show, however, that under these assumptions $f_n^{\#}\to 0$ compactly in U.

3. Find explicit formulas for a Riemann map $\mathbb{D}\to U$ when

(a) U is the strip $\{\,z\in\mathbb{C}\mid 0<\mathrm{Im}(z)<1\,\}$

(b) U is the half-strip $\{\,z\in\mathbb{C}\mid 0<\mathrm{Im}(z)<1 \quad \text{and} \quad \mathrm{Re}(z)>0\,\}$

4. Let $k\geq 2$ be an integer and $\omega=e^{2\pi i/k}$. Suppose $U\subsetneq\mathbb{C}$ is a simply connected domain such that $z\in U$ if and only if $\omega z\in U$ (thus, U has a rotational symmetry by the angle $2\pi/k$). Consider a Riemann map $f:\mathbb{D}\to U$ with $f(0)=0$. If $f(z)=\sum_{n=1}^{\infty}a_n z^n$, prove that

$$a_n=0 \qquad \text{if } n\not\equiv 1 \pmod{k}.$$

As an example, if U has a 180° symmetry about the origin, its Riemann map can contain only the odd powers of z. (Hint: Show that $f(\omega z)=\omega f(z)$ for all $z\in\mathbb{D}$.)

5. Let $U\subsetneq\mathbb{C}$ be a simply connected domain with $p\in$. Define the **conformal radius of U at p** to be $(U,p)=|f'(0)|$, where f is any Riemann map of U sending 0 to p.

(a) Verify that the definition of (U,p) is independent of the choice of f.

(b) Let $V\subsetneq U$ be a simply connected domain with $\phi:U\to V$ be holomorphic with $\phi(0)=q$, show that

$$|\phi'(p)|\leq\frac{(V,q)}{(U,p)}\,,$$

with equality if and only if ϕ is a conformal isomorphism.