MAT536 Homework 8

Due Wednesday, April 5

1. Find an element of $\mathrm{Aut}(\widehat{\mathbb{C}})$ which maps the upper half-disk $U = \mathbb{H} \cap \mathbb{D} = \{ z \in \mathbb{D} \mid \mathrm{Im}(z) > 0 \}$ onto the first quadrant $V = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0,\ \mathrm{Im}(z) > 0\}$. Use this map to give the explicit formula for a biholomorphism $f : U \to \mathbb{D}$.

2. Determine the automorphism group of the thrice-punctured sphere $\widehat{\mathbb{C}} \smallsetminus \{ 0, 1, \infty \}$. (Hint: Every homeomorphism of $\widehat{\mathbb{C}} \smallsetminus \{0, 1, \infty\}$ extends uniquely to a homeomorphism of $\widehat{\mathbb{C}}$.)

3. Let $p_1, \dots, p_n \in \mathbb{D}$ and $|\alpha| = 1$. The rational function

$$B(z) = \alpha \prod_{k=1}^{n} \varphi_{p_k}(z) = \alpha \prod_{k=1}^{n} \frac{z - p_k}{1 - \overline{p_k} z}$$

is called a **finite Blaschke product**.

(a) Show that B satisfies $B(1/\bar{z}) = 1/\overline{B(z)}$ for all z. In other words, B commutes with the reflection $z \mapsto 1/\bar{z}$ across the unit circle.

(b) Verify that $B(\mathbb{T}) \subseteq \mathbb{T}$, $\quad B(\mathbb{D}) \subseteq \mathbb{D}$, $\quad$ and that $\quad B(\widehat{\mathbb{C}} \smallsetminus \overline{\mathbb{D}}) \subseteq \widehat{\mathbb{C}} \smallsetminus \overline{\mathbb{D}}$.

(c) If $f : \mathbb{D} \to \mathbb{D}$ is any holomorphic function which vanishes at $p_1, \dots, p_n$, show that

$$|f(z)| \leq |B(z)| = \prod_{k=1}^{n} \left| \frac{z - p_k}{1 - \overline{p_k} z} \right| \qquad \text{for all } z \in \mathbb{D}.$$

What can be said about f if equality holds for some $z \in \mathbb{D} \smallsetminus \{ p_1, \dots, p_n \}$?
(Hint: imitate the proof of the Schwarz lemma, which is the special case of this when $B(z) = \varphi_0(z) = z$.)

4. Let f and g be non-identity elements of the group of Möbius maps. If f and g have the same set of fixed points, show that they commute, that is $f \circ g = g \circ f$. Conversely, if f and g commute, show that either they have the same set of fixed points or f and g are involutions (i.e. $f \circ f = g \circ g = \mathrm{id}$) which swap each others fixed points (such as the pair $f(z) = -z$ and $g(z) = 1/z$).

5. If $f : \mathbb{D} \to \mathbb{D}^*$ is holomorphic, prove that $|f'(0)| \leq 2/e$.