MAT536 Homework 7

Due Wednesday, March 22 (but do it before)

1. *(A generalized argument principle)*
Let γ be a positively oriented Jordan curve in a domain U such that $D = \text{int}(\gamma) \subset U$. Suppose f is meromorphic in U with no zeros or poles on $\{\gamma\}$. Let $\{z_j\}_{j=1}^m$ and $\{p_j\}_{j=1}^n$ denote the zeros and poles of f in D. If $\phi \in \mathcal{O}(U)$, show that

$$\frac{1}{2\pi i}\int_\gamma \phi(z)\frac{f'(z)}{f(z)}\,dz = \sum_{j=1}^m \deg(f, z_j)\,\phi(z_j) - \sum_{j=1}^n \deg(f, p_j)\,\phi(p_j).$$

The standard argument principle corresponds to the constant function $\phi = 1$.

2. Suppose that $f(z)$ is holomorphic for $|z| < 2$, let $\mathbb{T}$ be the positively oriented unit circle. If

$$\frac{1}{2\pi i}\int_\mathbb{T} \frac{f'(z)}{f(z)}\,dz = 2 \quad , \quad \frac{1}{2\pi i}\int_\mathbb{T} \frac{z\,f'(z)}{f(z)}\,dz = 1 \quad \text{, and} \quad \frac{1}{2\pi i}\int_\mathbb{T} \frac{z^2 f'(z)}{f(z)}\,dz = \frac{5}{9} \quad ,$$

determine the zeros of f within the unit disk $\mathbb{D}$. (The previous problem might be helpful.)

3. Counting multiplicities, how many roots of the equation

$$z^7 - 2z^5 + 6z^3 - z + 1 = 0$$

lie in each of the three sets $\mathbb{D}$, $\mathbb{D}_2(0) \setminus \overline{\mathbb{D}}$, and $\mathbb{C} \setminus \overline{\mathbb{D}}_2(0)$?

4. *(Inverse Function Theorem)*
Let γ be a positively oriented Jordan curve in a domain U so that $D = \text{int}(\gamma) \subset U$. Suppose $f \in \mathcal{O}(U)$ is injective in D, that is, $f : D \to f(D)$ is a biholomorphism. Show that

$$f^{-1}(w) = \frac{1}{2\pi i}\int_\gamma \frac{\zeta\, f'(\zeta)}{f(\zeta) - w}\,d\zeta$$

for every $w \in f(D)$.

5. Show that a Möbius function f satisfies $f(\mathbb{D}) = \mathbb{D}$ if and only if

$$f(z) = \frac{az + b}{\bar{b}z + \bar{a}} \qquad \text{for} \quad a, b \in \mathbb{C} \quad \text{with} \quad |a|^2 - |b|^2 = 1 \quad .$$