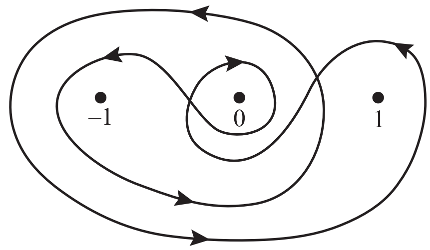
MAT536 Homework 6
Due Wednesday, March 8

1. Use the residue theorem to compute

(a) $\displaystyle\int_{\mathbb{T}} \frac{e^{3z}-e^{-2z}}{z^4}\,dz$

(b) $\displaystyle\int_{\gamma} \frac{1}{z^3(z^2-1)}\,dz$,



where $\mathbb{T} = \mathbb{T}_1(0)$ is the positively oriented unit circle, and γ is the curve shown above.

2. This exercise outlines an application of residues in evaluating integrals of rational functions over the real line.

(a) Let P and Q be polynomials with real coefficients, with $\deg Q \geq \deg P + 2$, and $Q(x) \neq 0$ for all $x \in \mathbb{R}$. Show that

$$\int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)}\,dx = 2\pi i \sum_j \operatorname{res}\left(\frac{P}{Q}, z_j\right),$$

where the sum is taken over all zeros z_j of Q in the upper half-plane.

(b) Compute the integral $\displaystyle\int_{-\infty}^{+\infty} \frac{x^2}{x^4+1}\,dx$.

Hint for the first part: Let γ_a be the closed curve consisting of the oriented interval $[-a,a]$ in $\mathbb{R}$ followed by the semicircle $|z| = a$ in the upper half-plane. Apply the residue theorem to compute $\int_{\gamma_a} P(z)/Q(z)\,dz$ for large $a > 0$, and show that the integral over the semicircle part tends to zero as $a \to +\infty$.

3. Write the Laurent series for the function

$$f(z) = \frac{z}{(z^2+4)(z-3)^2(z-4)} \quad \text{for} \quad \{z : 3 < |z| < 4\} .$$

4. Show that although the residue of a function $f(z)$ is not invariant under a local change of coordinates, the residue of the differential $f(z)\,dz$ is invariant.

That is, suppose ϕ maps a neighborhood U of p biholomorphically to a neighborhood V of $q = \phi(p)$. Then for any $f \in \mathcal{O}(V)$ with an isolated singularity at q, we have

$$\operatorname{res}(f,q) = \operatorname{res}\big((f\circ\phi)\phi', p\big) .$$

5. If infinity is a fixed point of a holomorphic map f, relate the index of the fixed point at ∞ to the residue by showing

$$\operatorname{ind}(f,\infty) = \operatorname{res}\left(\frac{f(z)}{z(z-f(z))}, \infty\right) .$$