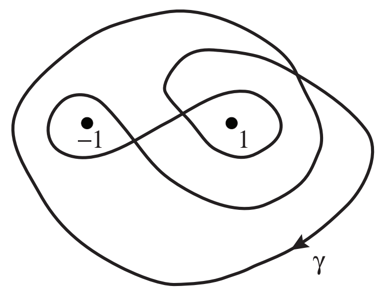
MAT536 Homework 5

Due Wednesday, February 29

1. Let $U = \mathbb{C} \setminus \{-1, 1\}$ and $f \in \mathcal{O}(U)$ so that $\int_\eta f(z)\,dz = a$ and $\int_\xi f(z)\,dz = b$, where η is the circle of radius 1 around 1, and ξ is the circle of radius 1 around -1 (both positively oriented).

Express $\int_\gamma f(z)\,dz$ in terms of a and b, where γ is the closed curve shown below:



2. Recall that the complex cosine and sine are the entire functions defined by

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}\,.$$

For each of the functions below, find their isolated singularities and determine their type (and order, if they are poles).

(a) $\dfrac{1 - \cos z}{z^2}$

(b) $\dfrac{z}{\sin z}$

(c) $\sin(1/z)$

3. Let f be a holomorphic function on a punctured open unit disk $\mathbb{D}^* = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$. In each case, prove or give a counter-example.

(a) If f has a removable singularity at $z = 0$, so does e^f .

(b) If f has a pole at $z = 0$, so does e^f .

(c) If f has an essential singularity at $z = 0$, so does e^f .

4. Suppose $f \in \mathcal{O}\left(\overline{\mathbb{D}} \setminus \{i\}\right)$, that is, f is holomorphic on the closed unit disk except for a singularity at $z = i$. Let $\sum_{n=0}^{\infty} a_n z^n$ be the power series for f on $\mathbb{D}$. Prove that if f has a pole at $z = i$, then

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = i\,.$$

5. Suppose f is entire and $f(z) \to \infty$ as $z \to \infty$. Show that f must be a polynomial.