MAT536 Homework 4
Due Wednesday, February 22

1. Consider the curves $\gamma, \eta : [0, 2\pi] \to \mathbb{C}$ given by $\gamma(t) = \sin(t)\, e^{it}$, $\eta(t) = \sin(t)\, e^{2it}$.
Sketch the image of each curve (indicating orientation), and find the values of the winding numbers $W(\gamma, \cdot)$ and $W(\gamma, \cdot)$ in each component of $\mathbb{C} \setminus \{\gamma\}$ and $\mathbb{C} \setminus \{\eta\}$, respectively.

2. Show that the function $f(z) = z$ has no holomorphic square root in the unit disk $\mathbb{D}$, i.e., there is no $f \in \mathcal{O}(\mathbb{D})$ such that $f^2(z) = z$ for $|z| < 1$.
By contrast, show that there is a unique $f \in \mathcal{O}(\mathbb{D}_1(1))$ which satisfies $f^2(z) = z$ and $f(1) = 1$. Writing $\sqrt{z}$ for $f(z)$, verify the expansion

$$\sqrt{z} = 1 + \frac{1}{2}(z-1) - \frac{1}{8}(z-1)^2 + \frac{1}{16}(z-1)^3 - \frac{5}{128}(z-1)^4 + \cdots$$

for $|z-1| < 1$.

3. Let $f \in \mathcal{O}(U)$, with $U \subset \mathbb{C}$ a simply connected domain. Suppose $f^{-1}(0) = \{p_1, \ldots, p_n\}$, and also for each k between 1 and n, $\operatorname{ord}(f, p_k)$ is an even integer. Show that f has a holomorphic square root in U.

4. Let the image of γ be a positively oriented smooth Jordan curve in $\mathbb{C}$. What quantity does

$$\frac{1}{2i} \int_\gamma \bar{z}\, dz$$

measure? Does the meaning change if γ has a finite number of self-intersections?
Hint: write as a pair of line integrals and use Green's theorem.

5. Is there a closed curve $\gamma : [0,1] \to \mathbb{C}$ so that the winding number function $p \mapsto W(\gamma, p)$ defined on $\mathbb{C} \setminus \{\gamma\}$ takes on

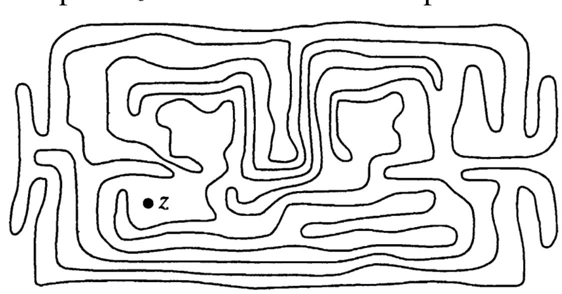
(a) only the values 0,1,2,3?

(b) only the values 0,1,3?

(c) every integer value?

(For each answer above, give an explicit example or a proof why it cannot occur. "Explicit" doesn't necessarily mean a formula.)

6. *(You don't have to hand in this one)*[†]
In the image below, is the point z inside or outside? Explain.



[†]taken from *Complex Analysis by Theodore Gamelin, p.251*