MAT536 Homework 3

Due Wednesday, February 15

1. Suppose $f, g \in \mathcal{O}(\mathbb{D})$ and neither f nor g has a zero in $\mathbb{D}$. If

$$\frac{f'(z)}{f(z)} = \frac{g'(z)}{g(z)} \qquad \text{for } z = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots$$

what relation between f and g must hold?

2. Suppose $U \subset \mathbb{C}$ is a domain and $f, g \in \mathcal{O}(U)$. If $|f(z)|^2 + |g(z)|^2 = 1$ for all $z \in U$, show that both f and g are constant.

3. Suppose $f \in \mathcal{O}(\mathbb{C})$ and there is a constant $C > 0$ such that

$$\int_0^{2\pi} |f(re^{it})| \, dt \le C r^{3/2} \qquad \text{for all } r > 0.$$

Prove that $f = 0$ in $\mathbb{C}$. What property of the constant $3/2$ is used here?
(Hint: Use Cauchy's integral formula for the higher derivatives $f^{(n)}(0)$.)

4. Suppose $U \subset \mathbb{C}$ is a bounded domain, $f : \overline{U} \to \mathbb{C}$ is continuous, $f \in \mathcal{O}(U)$, and $|f|$ is constant on ∂U. If $f \neq 0$ in U, show that f must be constant in U.
Give an example showing that the assumption $f \neq 0$ in U is essential.

5. Let $X = \mathbb{C} \setminus \{-1, 1\}$ be the twice-punctured plane, and let

$$\gamma(t) = e^{2\pi i t} - 1 \qquad \eta(t) = -\gamma(t) = 1 - e^{2\pi i t} .$$

Show that the closed curves $\gamma \cdot \eta$ and $\eta \cdot \gamma$ are freely homotopic in X, but are not homotopic in X with basepoint 0.