MAT536 Homework 2

Due Wednesday, February 8

As usual, $\mathbb{D}_r(p)$ is the disk of radius r centered at p, $\mathbb{D} = \mathbb{D}_1(0)$ is the unit disk; $\mathbb{T}_r(p)$ is the positively oriented circle of radius r centered at p, and $\mathbb{T} = \mathbb{T}_1(0)$ is the unit circle (positively oriented).

1. Use Cauchy's integral formula to evaluate

(a) $\displaystyle\int_{\mathbb{T}} \frac{dz}{4z^2+1}$ (Hint: use partial fractions)

(b) $\displaystyle\int_0^{2\pi} \frac{dt}{|e^{it}-a|^2}$ $(a \in \mathbb{C},\ |a| \neq 1)$

Hint: Write the integrand as $\dfrac{1}{(e^{it}-a)(e^{-it}-\bar{a})} = \dfrac{e^{it}}{(e^{it}-a)(1-\bar{a}e^{it})}$ and set $z = e^{it}$ to express the integral as a complex integral over $\mathbb{T}$. You should consider the cases $|a| < 1$ and $|a| > 1$ separately.

2. Use Cauchy's integral formula in a disk for higher derivatives to evaluate $\displaystyle\int_{\mathbb{T}} z^k e^z \, dz$ where k is an integer.

3. Let $f(z)$ be entire. Prove or give a counterexample.

(a) Suppose $\displaystyle\lim_{|z|\to\infty} f(z) = 0$. Then f must be constant.

(b) Suppose there is a sequence $\{z_n\}$ with $\displaystyle\lim_{n\to\infty} z_n = \infty$ and $f(z_n) = 0$. Then f must be constant.

4. Let $f(z)$ be entire and satisfy $|f(z)| < |z|^n$ for some n and all $|z|$ sufficiently large. Prove that f must be a polynomial.

5. A function $f : \mathbb{C} \to \mathbb{C}$ is called *doubly periodic* if there are non-zero complex numbers α, β, with $\alpha/\beta \notin \mathbb{R}$, such that

$$f(z+\alpha) = f(z+\beta) = f(z) \qquad \text{for all } z \in \mathbb{C}.$$

Show that every doubly periodic entire function is constant.