MAT536 Homework 1
Due Wednesday, February 1

1. This exercise develops several useful formulas in complex-variable notation. Consider the differential operators

$$f_z = \frac{1}{2}(f_x - i f_y) \qquad \text{and} \qquad f_{\bar{z}} = \frac{1}{2}(f_x + i f_y)$$

acting on smooth functions $f : U \to$. Recall that for $f = u + iv$, the notation $\bar{f}$ is used for the conjugate function $u - iv$.

(a) Verify the product rules

$$(f\,g)_z = f_z\,g + f\,g_z$$
$$(f\,g)_{\bar{z}} = f_{\bar{z}}\,g + f\,g_{\bar{z}}.$$

(b) Verify the chain rules

$$(f \circ g)_z = (f_z \circ g)\,g_z + (f_{\bar{z}} \circ g)\,\overline{g}_z$$
$$(f \circ g)_{\bar{z}} = (f_z \circ g)\,g_{\bar{z}} + (f_{\bar{z}} \circ g)\,\overline{g}_{\bar{z}}.$$

(c) Show that

$$\overline{f}_z = \overline{(f_{\bar{z}})} \qquad \text{and} \qquad \overline{f}_{\bar{z}} = \overline{(f_z)}.$$

(d) Let J_f be the Jacobian determinant of f, viewed as a map $U \to \mathcal{R}^2$:

$$J_f = \det Df = \det \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}.$$

Verify that $J_f = |f_z|^2 - |f_{\bar{z}}|^2$. In particular, if $f \in \mathcal{O}(U)$, then $J_f = |f'|^2$.

(e) Verify that

$$\Delta f = 4 f_{z\bar{z}} = 4 f_{\bar{z}z},$$

where $\Delta f = f_{xx} + f_{yy}$ is the usual Laplacian of f.

(f) Show that if $f \in \mathcal{O}(U)$, then $\Delta |f|^2 = 4|f'|^2$.

2. Let $f = u + iv \in \mathcal{O}(U)$. Show that

$$|f'| = \|\nabla u\| = \|\nabla v\| \qquad \text{in } U,$$

where ∇u is the gradient of u and $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^2$. At every point where $f' \neq 0$, find a simple geometric interpretation for the Cauchy-Riemann equations in terms of the vectors ∇u and ∇v at that point.

3. Determine the radius of convergence of the power series

$$\sum_{j=1}^{\infty} j\,z^j \qquad \text{and} \qquad \sum_{j=1}^{\infty} j^2\,z^j.$$

What familiar functions of z do these series represent in their disk of convergence? Give formulas.