Power,Miguel,9point

Wednesday, November 18, 2020 5:22 PM

LAST TIME SAW THAT IF P IS A POINT NOT ON CIRCLE $\mathcal{C}$ WITH CENTER O, RAD $= r$, ℓ ANY LINE CONTAINING P INTERSECTING $\mathcal{C}$ AT A AND B ($A=B$ OK) THEN

$$\boxed{\mathrm{Pow}(P,\mathcal{C}) = |PA|\cdot|PB| = \big|\,|OP|^2 - r^2\big|}$$

ABS VAL

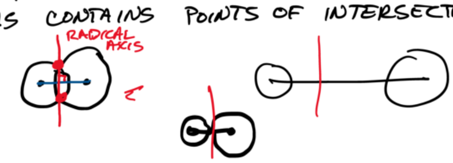
THE POWER OF P WITH RESPECT TO $\mathcal{C}$

GIVES RISE TO

DEF GIVEN CIRCLES $\mathcal{C}_1, \mathcal{C}_2$ WITH DISTINCT CENTERS O_1, O_2 & RADII r_1, r_2. DEFINE THE RADICAL AXIS OF $\mathcal{C}_1, \mathcal{C}_2$ IS THE SET OF ALL P WITH $\mathrm{Pow}(P,\mathcal{C}_1) = \mathrm{Pow}(P,\mathcal{C}_2)$

THM THE RADICAL AXIS IS A LINE PERPENDICULAR TO $\overleftrightarrow{O_1O_2}$

IF $\mathcal{C}_1$ INTERSECTS $\mathcal{C}_2$ THEN THE RADICAL AXIS CONTAINS POINTS OF INTERSECTION.



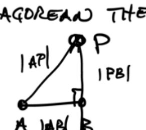
PROOF USES LEMMA:

GIVEN A & B AND A CONSTANT β. THE SET $\{P : |PA|^2 - |PB|^2 = \beta\}$ IS A LINE PERPENDICULAR TO $\overleftrightarrow{AB}$

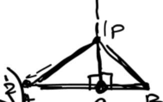
IDEA OF OF PROOF:
(CAN ASSUME WLOG $\beta > 0$)

- IF $\beta = |AB|^2$ THEN BY PYTHAGOREAN THEOREM

$$\beta = |PA|^2 - |PB|^2 = |AB|^2$$



- IF $\beta < |AB|^2$



$$|PA|^2 - |PB|^2$$
$$= (|PC|^2 + |CA|^2) - (|PC|^2 + |CB|^2)$$
$$= |AC|^2 - |BC|^2 = \beta$$

IF $\beta > |AB|^2$

$$|PA|^2 - |PB|^2 = (|PC|^2 + |CA|^2) - (|PC|^2 + |CB|^2)$$
$$|AC|^2 - |BC|^2$$

SO LINE IS ALWAYS $\perp$ TO $\overline{AB}$ ☒

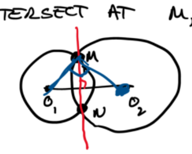
PF OF THM O_1, O_2 CENTERS $\quad O_1 \neq O_2$
$r_1 \ r_2$ RADII

SINCE $\mathrm{Pow}(P,\mathcal{C}_1) = \mathrm{Pow}(P,\mathcal{C}_2)$

$$|O_1P|^2 - r_1^2 = |O_2P|^2 - r_2^2$$
$$|O_1P|^2 - |O_2P|^2 = r_1^2 - r_2^2 = \text{const}, \beta$$

SO ANY SUCH P IS ON LINE $\perp$ TO $\overline{O_1O_2}$

IF $\mathcal{C}_1, \mathcal{C}_2$ INTERSECT AT M, N

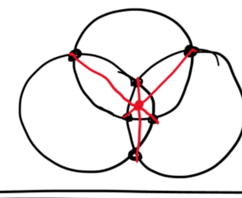


SO USE HOMEWORK PROB TO SEE THIS -

APPLICATION:

SPOZE 3 CIRCLES MEETING IN PAIRS AT 3 PTS

LINES JOINING THESE PTS ARE CONCURRENT



MIQUEL POINT

THM: LET D, E, F BE POINTS ON $\overleftrightarrow{AB}, \overleftrightarrow{BC}$ AND $\overleftrightarrow{AC}$. THEN CIRCUMCIRCLES OF $\triangle ADF$, $\triangle DBE$, AND $\triangle CEF$ HAVE A POINT M IN COMMON (THE MIQUEL POINT)

AUGUSTE MIQUEL 1838

PF/ LET $\mathcal{C}_{DAF}$ & $\mathcal{C}_{DBE}$ BE THE CIRCUMCIRCLES OF $\triangle DAF$ & $\triangle DBE$,

LET M BE THEIR POINT OF INTERSECTION ($\neq D$)

TO SHOW RESULT, MUST SHOW THAT $\square EMFC$ IS CYCLIC

IF $\angle A$ IS RIGHT ANGLE THEN $M = E$ DONE SO $\square EMFC = \triangle EMC$

CASE $\angle A$ IS ACUTE

$\measuredangle BEM = \measuredangle MDA$ SINCE $\square BEMD$ IS CYCLIC & $\angle MDA$ IS EXTERIOR ANGLE, OPPOSITE $\angle BEM$

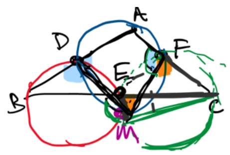
SIMILARLY $\measuredangle MDA = \measuredangle MFC$ SINCE $\angle MFC$ EXT TO $\square ADME$ OPPOSITE TO $\measuredangle MDA$

~~[illegible]~~LY, $\measuredangle BEM = \measuredangle MFC$

& $\square EMFC$ IS CYCLIC. ✓

A

IF $\angle A$ IS OBTUSE & SO M IS OUTSIDE $\triangle ABC$



AS BEFORE, LET M BE THE INTERSECTION OF CIRCUMCIRCLES OF $\triangle BDM$ & $\triangle FAD$.

& SHOW $\square MEFC$ IS CYCLIC. WANT $\measuredangle MEC = \measuredangle MFC$

TO DO THAT, CAN SHOW $\measuredangle MEB = \measuredangle MFA$ (TRUE SINCE SUPPLEMENTARY (IE ADD TO $180°$))

BUT $\measuredangle MFA = \measuredangle MDB$ (SINCE $\measuredangle MDB$ EXTERNAL TO QUAD MDAF)

AND $\measuredangle MDB = \measuredangle MEB$ SINCE OPP EXTERNAL. ☒

NINE POINT CIRCLE $\triangle ABC$

RECALL G = CENTROID = INTERSECTION OF MEDIANS

H = ORTHOCENTER = INTERSECT OF ALTITUDES

O = CIRCUMCENTER = INTERSECT OF $\perp$ BSCT OF SIDES

IF $G=O$, THEN $\triangle ABC$ IS EQUILATERAL $G=H=O$.

ASSUME $G \neq O$. $\overline{HO}$ IS EULER SEGMENT

THM: LET Δ BE THE DILATION BY $\frac{1}{2}$ CENTERED AT H. LET $\mathcal{C}$ BE THE CIRCUMCIRCLE OF $\triangle ABC$ (CENTER $= O$) LET $\mathcal{N} = \Delta(\mathcal{C})$

THEN: • THE CENTER OF $\mathcal{N}$ IS THE MIDPOINT OF $\overline{HO}$

• $\mathcal{N}$ CONTAINS

(a) THE 3 FEET OF ALTITUDES

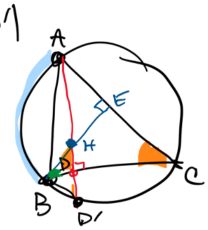
(m) THE 3 MIDPOINTS OF SIDES

(h) THE MIDPOINTS OF $\overline{AH}$, $\overline{BH}$ AND $\overline{CH}$

FEURBACH'S THM: INCIRCLE OF $\triangle ABC$ IS TANGENT TO 9 PT CIRCLE. [WE WON'T PROVE THIS]

PROOF NEEDS A LEMMA

LET $\overline{AD}$ BE AN ALTITUDE OF $\triangle ABC$ (FOOT D), AND LET $\overrightarrow{AD}$ INTERSECT THE CIRCUMCIRCLE AT D'. H IS THE ORTHOCENTER OF $\triangle ABC$. THEN $|HD| = |DD'|$



PF/ • OBSERVE THAT $\measuredangle BHD = \measuredangle ACB$

SINCE $\triangle BEC$ IS RIGHT

SO $\measuredangle BBE + \measuredangle BCE = 90°$

ALSO $\triangle BHD$ IS RIGHT, SO

$\measuredangle CBE + \measuredangle BHD = 90° \Rightarrow \measuredangle BHD = \measuredangle BCE$ ($\measuredangle ACB$)

• BUT $\angle ACB$ & $\angle AD'B$ BOTH SUBTEND $\overline{AB}$

SO $\triangle BD'H$ IS ISOSCELES.

SO $\overline{BD}$ IS $\perp$ BISECTOR OF $\overline{HD}$ ☒

Pf OF 9-PT CIRCLE:

(h) IMMEDIATE SINCE $\Delta(A) = H_A$ (SCALE TOWARDS H BY $\frac{1}{2}$)

H_A IS HALFWAY

(a) FOLLOWS FROM LEMMA & DILATION:

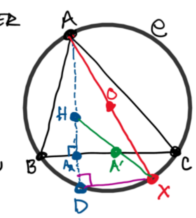
SINCE A_A LIES ON $\overrightarrow{AH}$

LEMMA SAY IT IS MIDPOINT OF $|HD|$ WHERE D IS ON CIRCUMCIRCLE

(m) TO SHOW THE RESULT ABOUT MIDPOINTS OF SIDES TAKES LONGER, BUT WE ARE OUT OF TIME. I WILL ADD THIS WHEN I POST THE NOTES.

↓ ADDED LATER ↓

RECALL THAT O IS THE CENTER OF THE CIRCUMCIRCLE $\mathcal{C}$. LET X BE THE OPPOSITE END OF THE DIAMETER FROM A.



DRAW $\overline{HX}$, AND LET THE INTERSECTION WITH $\overline{BC}$ BE A'.

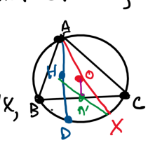
WE WANT TO SHOW THAT $|HA'| = |A'X|$, (THAT IS, THAT $\Delta(X) = A'$, SO A' LIES ON $\mathcal{N}$)

JOIN D (THE INTERSECTION OF THE ALTITUDE WITH $\mathcal{C}$) TO X. THEN $\measuredangle ADX = 90°$ SINCE $\overline{AX}$ IS A DIAMETER.

BOTH $\overline{DX}$ AND $\overline{BC}$ ARE PERPENDICULAR TO $\overleftrightarrow{AD}$ AND SO $\overleftrightarrow{DX} \parallel \overleftrightarrow{BC}$.

WE'VE SHOWN THAT A_A (THE FOOT OF THE ALTITUDE FROM A) IS THE MIDPOINT OF $\overline{HD}$, SO BY FTS, A' IS THE MIDPOINT OF $\overline{HX}$.

TO SEE THAT A' IS THE MIDPOINT OF $\overline{BC}$, CONSIDER $\triangle AHX$.



OBSERVE THAT $\overline{OA'}$ JOINS THE MIDPOINTS OF TWO SIDES OF $\triangle AHX$, AND SO $\overline{OA'} \parallel \overline{AH}$. BUT ALSO, $\overleftrightarrow{AH} \perp \overleftrightarrow{BC}$ WE KNOW $\overleftrightarrow{OA'}$ IS PERPENDICULAR TO $\overline{BC}$. SINCE WE HAVE A SEGMENT FROM THE CENTER O PERPENDICULAR TO CHORD $\overline{BC}$, $\overline{OA'}$ IS IN FACT THE PERPENDICULAR BISECTOR.

LASTLY, WE HAVE TO SHOW THAT THE CENTER N OF THE NINE-POINT CIRCLE IS THE MIDPOINT OF THE EULER SEGMENT $\overline{HO}$. BUT THIS IS IMMEDIATE, SINCE $N = \Delta(O)$ AND Δ SCALES DISTANCES FROM H BY $\frac{1}{2}$. ☒

IN ADDITION TO BEING TANGENT TO THE INCIRCLE AT ONE POINT AS STATED EARLIER, THE NINE-POINT CIRCLE $\mathcal{N}$ IS ALSO TANGENT TO EACH OF THE THREE EXCIRCLES.

HISTORICALLY (AND OFTEN IN OTHER PRESENTATIONS), THE 9-POINT CIRCLE IS NOT CONSTRUCTED VIA A DILATION OF THE CIRCUMCIRCLE, BUT VIA OTHER OF THE SIGNIFICANT POINTS.

IT HAS MANY OTHER PROPERTIES & NAMES (FEURBACH'S CIRCLE, EULER'S CIRCLE, TERQUEM'S CI CIRCLE, THE MEDIOSCRIBED CIRCLE, THE MID-CIRCLE, THE 12-POINT CIRCLE ....)

ITS DISCOVERY WAS A LONG PROCESS AND SIGNIFICA IN EARLY 19TH CENTURY GEOMETRY.