Even more circles

Monday, November 16, 2020 2:44 PM

LAST TIME, WE PROVED THE FIRST PART OF

THM:

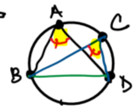
LET A, B, C, D BE FOUR POINTS.

(i) LET A & C BE ON THE SAME SIDE OF $\overleftrightarrow{BD}$

A, B, C, D CONCYCLIC $\iff$ $\measuredangle BAD = \measuredangle BCD$

(ii) LET A & C BE ON OPPOSITE SIDES OF $\overleftrightarrow{BD}$

A, B, C, D CONCYCLIC $\iff$ $\measuredangle BAD + \measuredangle BCD = 180°$



PF/ (i) WAS LAST TIME.

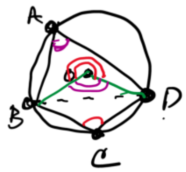
SO NOW (ii)

$\Rightarrow$ (ASSUME $\overset{\frown}{BAD}$ IS A MAJOR ARC.

$\measuredangle BAD = \frac{1}{2} \measuredangle$ CONVEX BOD

$\measuredangle BCD = \frac{1}{2} \measuredangle$ NONCONVEX BOD

$\measuredangle BAD + \measuredangle BCD = \frac{1}{2}(360°) = 180°$ ✓



$\Leftarrow$ SIMILAR TO PROOF FOR (i)

SPOSE ABCD NOT CONCYCLIC

EITHER C INTERIOR CIRCLE $\mathcal{C}$ DEFINED BY A, B, D OR EXTERIOR.

DO INTERIOR CASE.

EXTEND $\overrightarrow{BC}$ TO MEET $\mathcal{C}$ AT E

A & C ARE ON OPP. SIDES OF $\overleftrightarrow{BD}$, C & E SAME

& SO A & E ON OPP SIDES OF $\overleftrightarrow{BD}$

$\angle BCD$ IS EXTERIOR TO $\triangle ECD$, SO

$\measuredangle BCD > \measuredangle BED$

BUT ALSO, $\measuredangle BAD + \measuredangle BED = 180°$

SO $\measuredangle BAD + \measuredangle BCD > 180°$ ☒

IN SOME TEXTS, THIS IS

"THE VERTICES OF A QUADRILATERAL LIE ON A CIRCLE $\iff$ OPPOSITE ANGLES SUM TO 180°"

ACTUALLY FALSE AS STATED

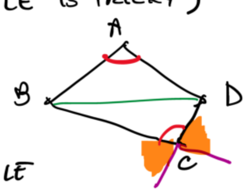
$\measuredangle BCD + \measuredangle BAD = 180°$

TO MAKE IT CORRECT! HAVE TO ADD "INTERIOR" TO ANGLE. BETTER IS TO SAY CONVEX QUADRILATERAL

(DEFINING INTERIOR ANGLE IS TRICKY)

SPOSE $\square ABCD$ HAS A & C ON OPP SIDE OF B & D.

CAN DEFINE EXTERIOR ANGLE



EXT ANGLE AT C IS EITHER OF THE TWO EXTERIOR ANGLES OF $\triangle BDC$ AT C.

SINCE MEASURE IS SAME CAN SAY "THE EXTERIOR ANGLE AT C"

COR: GIVEN $\square ABCD$ WITH A & C ON OPP SIDES OF $\overleftrightarrow{BD}$

THE EXTERIOR ANGLE AT A = OPPOSITE INTERIOR ANGLE AT C

$\iff$ A, B, C, D ARE CONCYCLIC

$\square ABCD$ IS A CYCLIC QUADRILATERAL

OR INSCRIBED QUAD. IN $\mathcal{C}$

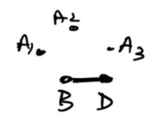
OR CHORDAL QUADRILATERAL

GENERALIZE THIS THEOREM:

THM: GIVEN B, D AND $A_1, A_2, \ldots, A_n$ BE POINTS IN THE SAME HALFPLANE OF $\overleftrightarrow{BD}$

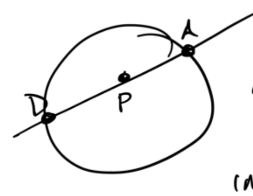
$B, D, A_1, A_2, \ldots, A_n$ ARE CONCYCLIC

$\iff$ $\measuredangle BA_iD$ ARE EQUAL FOR ALL i

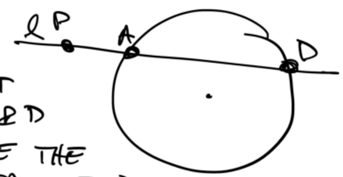


PF: ONE OF THE THINGS YOU GET TO DO FOR FINAL.

GIVEN A CIRCLE $\mathcal{C}$ WITH A POINT P NOT ON $\mathcal{C}$ AND A LINE ℓ CONTAINING P



LET A & D BE THE INTERSECTION OF ℓ WITH $\mathcal{C}$



$|PA| \cdot |PD|$ DOESN'T DEPEND ON ℓ.

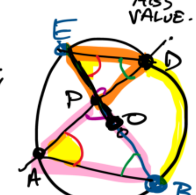
THM: LET $\mathcal{C}$ BE A CIRCLE OF RADIUS r CENTER O. LET P BE A POINT NOT ON $\mathcal{C}$.

LET ℓ BE A LINE CONTAINING P INTERSECTING $\mathcal{C}$ AT A & D

THEN $|PA| \cdot |PD| = |r^2 - |OP|^2|$ ← ABS VALUE.

Pf/

CASE i) P IS INSIDE $\mathcal{C}$



LET THE DIAMETER CONTAINING P INTERSECT $\mathcal{C}$ AT E & B

SINCE $\angle E$ AND $\angle A$ SUBTEND $\overset{\frown}{BD}$, $\measuredangle E = \measuredangle A$. $\measuredangle D = \measuredangle B$

ALSO SINCE SUBTEND $\overset{\frown}{EA}$

$\triangle PDE \sim \triangle PBA$ SO $\dfrac{|PA|}{|PE|} = \dfrac{|PB|}{|PD|}$

IE $|PA| \cdot |PD| = |PE| \cdot |PB|$

IE $|PA|\cdot|PD| = |PE|\cdot|PB|$

SINCE DIAMETER CAN SAY MORE

$\overline{BE}$ IS DIAMETER, SO $|BE| = 2r$

SO $|PB| = |BO| + |OP| = r + |OP|$

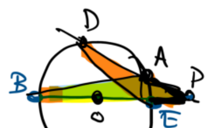
& $|PE| = |OE| - |OP| = r - |OP|$

$$|PB|\cdot|PE| = r^2 - |OP|^2 > 0$$

$$|PA|\cdot|PD|$$

NOW LET P BE EXTERIOR TO $\mathcal{C}$

AS BEFORE, LET B & E BE INTERSECTION OF $\mathcal{C}$ WITH $\overleftrightarrow{OP}$ WITH $B*O*E*P$



$\measuredangle B = \measuredangle D$ CUZ SUBTEND $\overset{\frown}{AE}$

SO $\Delta PED \sim \Delta PAB$

SO $\dfrac{|PA|}{|PE|} = \dfrac{|PB|}{|PD|}$ IE $|PA|\cdot|PD| = |PB|\cdot|PE|$

AS BEFORE $|PB| = |PO| + |OB| = |OP| + r$

$|PE| = |PO| - |OE| = |OP| - r$

$|PA|\cdot|PD| = |PB|\cdot|PE| = |OP|^2 - r^2$ ∎

THE CONVERSE HOLDS TOO:

THM LET ℓ_1 & ℓ_2 BE TWO LINES INTERSECTING AT P, WITH A, B, D, E SO THAT $A*P*D$ ON ℓ_1, $B*P*E$ ON ℓ_2.

$|PA|\cdot|PD| = |PB|\cdot|PE| \Rightarrow A, B, D, E$ CONCYCLIC

PF/ A & E ARE ON THE SAME SIDE OF $\overleftrightarrow{BD}$

(OBSERVE P & A ARE ON SAME SIDE OF $\overleftrightarrow{DB}$ SINCE $A*P*D$ IF $\overline{BD} \cap PA$, $\overleftrightarrow{BD} = \overleftrightarrow{PA}$ COS 2 PTS.

ALSO P & E IN SAME HALF, SO A, P, E " " " "

$\measuredangle APB = \measuredangle EPD$ (VERTICAL)

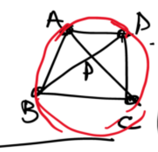
ALSO $\dfrac{|PA|}{|PE|} = \dfrac{|PB|}{|PD|}$ BY HYPOTHESIS, SO $\Delta PAB \sim \Delta PED$

BUT $\measuredangle A = \measuredangle E$. SO APPLY THM THAT IF $\measuredangle A$, $\measuredangle E$ ON SAME SIDE OF $\overline{BD}$ CONCYCLIC.

A, B, D, E ∎

REPHRASE AS

COR LET A, B, C, D BE A QUADRILATERAL WITH P BEING INTERSECTION OF DIAGONALS



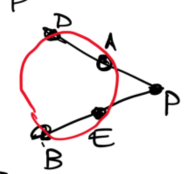
$\square ABCD$ IS CYCLIC $\iff |PA|\cdot|PC| = |PB|\cdot|PD|$

THM LET ℓ_1, ℓ_2 INTERSECT AT P

$D*A*P$, $B*E*P$

IF $|PA|\cdot|PD| = |PE|\cdot|PB|$

THEN A, D, B, E CONCYCLIC



SHARE $\angle P$, MATCH UP RATIOS

$\Delta PAB \sim \Delta PED \Rightarrow \angle D = \angle E$

DONE.

THEOREM STILL WORKS $B = E$,

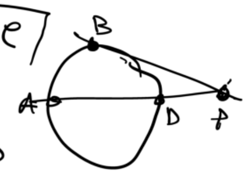
IE $\overleftrightarrow{PB}$ TANGENT TO $\mathcal{C}$

THM: LET P BE EXTERIOR TO $\mathcal{C}$

$\overleftrightarrow{PB}$ TANGENT TO $\mathcal{C}$ AT B

IF ANOTHER LINE ℓ CONTAINS P & INTERSECTS $\mathcal{C}$ AT A & D

THEN $|PA|\cdot|PD| = |PB|^2$



CONVERSELY IF A, D ON $\mathcal{C}$ SO THAT $\overleftrightarrow{AD}$ CONTAINS P EXTERIOR TO $\mathcal{C}$

AND B ON $\mathcal{C}$ SATISFIES $|PB|^2 = |PA|\cdot|PD|$

THEN $\overleftrightarrow{PB}$ IS TANGENT TO $\mathcal{C}$

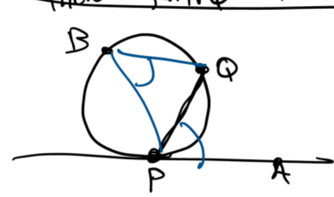
PROOF USE A LEMMA

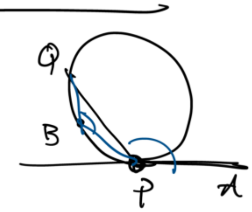
THM: LET $\mathcal{C}$ BE A CIRCLE TANGENT TO $\overleftrightarrow{PA}$ AT P

LET $\overline{PQ}$ BE A CHORD OF $\mathcal{C}$

B A POINT ON $\mathcal{C}$ NOT IN CONVEX $\angle APQ$

THEN $\measuredangle APQ = \measuredangle PBQ$





READ PROOFS IN BOOK.