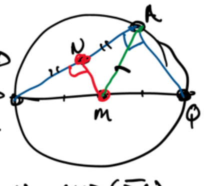
MoreCircles

Wednesday, November 11, 2020 5:55 PM

FROM LAST TIME:

THM: LET $\overline{PQ}$ BE A CHORD OF A CIRCLE $\mathcal{C}$ WITH $A \neq P, A \neq Q$ AND A ON $\mathcal{C}$.
$\measuredangle PAQ = 90° \iff \overline{PQ}$ IS A DIAMETER

WE ALREADY SHOWED IF $\overline{PQ}$ IS A DIAMETER THEN $\angle PAQ$ IS A RIGHT ANGLE.



$\Rightarrow$ SPOZE THAT $\overline{PQ}$ IS SOME CHORD AND THAT $\measuredangle PAQ = 90°$
WANT TO SHOW $\overline{PQ}$ IS DIAMETER (IE CENTER IS ON $\overline{PQ}$)

LET M BE THE MIDPOINT OF $\overline{PQ}$, $N = \text{MID}(\overline{PA})$
SINCE $|PM| = |MQ|$ AND $|PN| = |NA|$

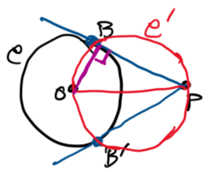
$$\frac{|MP|}{|PQ|} = \frac{|NP|}{|PA|} = \frac{1}{2} \Rightarrow \Delta MPN \sim \Delta QPA$$

SO $\overleftrightarrow{MN} \parallel \overleftrightarrow{AQ}$ SO $\overline{MN} \perp \overline{PA}$

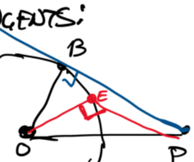
(SO $\Delta MAN \cong \Delta MPN \Rightarrow |MP| = |MA|$)
SO M IS CIRCUMCENTER OF ΔAPQ
$\Rightarrow$ CENTER OF $\mathcal{C}$.

$\Rightarrow \overline{PQ}$ IS A DIAMETER.

THM: GIVEN CIRCLE $\mathcal{C}$ AND P EXTERIOR TO $\mathcal{C}$ THEN THERE ARE EXACTLY TWO TANGENTS FROM P TO $\mathcal{C}$, AND THEY HAVE EQUAL LENGTH



- FIRST, SHOW THERE IS A TANGENT.
LET $\mathcal{C}'$ BE THE CIRCLE WITH DIAMETER $\overline{PO}$
(SINCE O IS INTERIOR TO $\mathcal{C}$ AND P IS EXTERIOR TO $\mathcal{C}$, THEN $\mathcal{C}$ INTERSECTS $\mathcal{C}'$)
LET B BE A POINT OF INTERSECTION.
SINCE B ON $\mathcal{C}'$, $\overline{OP}$ IS A DIAMETER
$\measuredangle OBP = 90°$.
$\overline{OB} \perp PB$, LINE $\overleftrightarrow{PB}$ IS TANGENT TO $\mathcal{C}$ AT B
- $\mathcal{C}$ AND $\mathcal{C}'$ IS SYMMETRIC WITH RESPECT TO A DIAMETER, SO REFLECT THRU $\overleftrightarrow{OP}$
$\Delta(B) = B'$, $\Delta(\mathcal{C}) = \mathcal{C}$, $\Delta(\mathcal{C}') = \mathcal{C}'$
SO $\overline{PB'}$ IS ANOTHER TANGENT TO $\mathcal{C}$
($B' \neq B$ SINCE THEY ARE IN OPPOSITE HALF PLANES WRT $\overleftrightarrow{OP}$)
- THERE CAN BE NO MORE TANGENTS:



IF THERE IS ANOTHER AT $E \neq B$:
(WLOG ASSUME SAME HALF AS B)
KNOW $\overline{OE} \perp$ TO $\overline{PE}$
$|OE| = |OB|$
BOTH SHARE HYP. $|OP|$ SO
$\Delta OEP \cong \Delta OBP$, SAME HALF PLANE
$\Rightarrow$ SAME Δ. SO $E = B$

IF A CHORD $\overline{PQ}$ IS NOT A DIAMETER THEN ONE OF THE HALFPLANES OF $\overleftrightarrow{PQ}$ CONTAINS CENTER O. THE ONE WITH O IS A MAJOR ARC $\widehat{PQ}$ } OPPOSITE ARCS
OTHER IS MINOR ARC $\widehat{PQ}$ } COMPLEMENTARY ARCS.

GIVEN ARC $\widehat{PQ}$ AND A IN OPPOSITE ARC

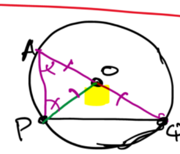
GIVEN $\measuredangle PAQ$ WITH P, A, Q ON $\mathcal{C}$.
THEN ARC $\widehat{PQ}$ (NOT CONTAINING A) IS INSCRIBED IN $\angle PAQ$

IF O IS THE CENTER OF $\mathcal{C}$,
$\angle POQ$ IS THE CENTRAL ANGLE SUBTENDED BY $\widehat{PQ}$

STARTING WITH $\widehat{PQ}$, THEN $\angle PAQ$ IS SUBTENDED BY $\widehat{PQ}$

ALL IS FINE FOR MAJOR OR MINOR ARCS. (MAJOR ARC HAS CENTRAL ANGLE BIGGER THAN 180°)

THM: LET $\widehat{PQ}$ BE AN ARC OF $\mathcal{C}$.
ALL ANGLES SUBTENDED BY $\widehat{PQ}$ HAVE THE SAME MEASURE.
THIS IS $\frac{1}{2}$ MEASURE OF CENTRAL ANGLE



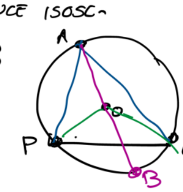
PROVE JUST FOR MINOR ARC.
CONSIDER 3 CASE

(i) O LIES ON A SIDE OF $\angle PAQ$

(ii) O LIES INTERIOR TO $\angle PAQ$

(iii) O LIES OUTSIDE $\angle PAQ$.

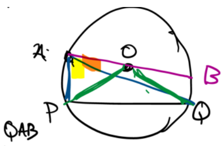
(i): SINCE O IS ON $\overline{AQ}$, $\overline{AQ}$ IS A DIAMETER
$\angle POQ$ IS EXTERNAL ANGLE TO ISOSC. TRIANGLE ΔPOA.
$\measuredangle POQ = \measuredangle PAO + \measuredangle OPA \underset{\text{SINCE ISOSC.}}{=} 2\measuredangle PAO$



(ii) EXTEND $\overrightarrow{AO}$ TO MEET $\mathcal{C}$ AT B
$\angle PAB$ IS ADJ TO $\angle BAQ$
$\measuredangle PAB + \measuredangle BAQ = \measuredangle PAQ$
FROM (i) $\measuredangle PAB = \frac{1}{2}\measuredangle POB$
$+ \quad \measuredangle BAQ = \frac{1}{2}\measuredangle BOQ$

$\measuredangle PAQ = \frac{1}{2}\measuredangle POQ$

(iii) EXTEND $\overrightarrow{AO}$ TO B



$\measuredangle PAQ + QAB = \measuredangle BAP$

$\measuredangle PAQ = \measuredangle BAP - \measuredangle QAB$

$\measuredangle POQ = \measuredangle BOP - \measuredangle BOQ$

BUT $\measuredangle BOP = 2\measuredangle BAP$ SINCE O ON $\overline{AB}$

$\measuredangle POQ = 2(\measuredangle BAQ - \measuredangle BOQ)$ ☺

COR: SPOZE $\widehat{PQ}$ AND $\widehat{P'Q'}$ ARE CONGRUENT ARCS ON $\mathcal{C}, \mathcal{C}'$ (so $\mathcal{C} \cong \mathcal{C}'$) THEY SUBTEND CONGRUENT ANGLES ON $\mathcal{C}, \mathcal{C}'$ AND CONGRUENT CENTRAL ANGLES

CONCYCLIC POINTS

KNOW ALREADY THAT 3 PTS (NONCOLLINEAR POINTS) DETERMINE A (UNIQUE) CIRCLE.

WHAT ABOUT 4 POINTS?

(ANALOGY TO CONCURRENT LINES)

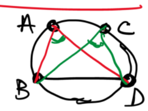
LOOK AT Δ, INTERSECTION OF $\perp$ BISECTORS GIVES CENTER... = CIRCUMCENTER

THM: LET A, B, C, D BE FOUR POINTS

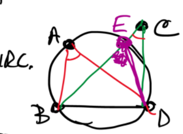
(i) IF A & C ARE ON SAME SIDE OF $\overleftrightarrow{BD}$
A, B, C, D LIE ON SAME CIRCLE (ARE CONCYCLIC) $\iff \measuredangle BAD = \measuredangle BCD$.

(ii) IF A & C ARE ON OPP. SIDES OF $\overleftrightarrow{BD}$
A, B, C, D CONCYCLIC $\iff \measuredangle BAD + \measuredangle BCD = 180^\circ$

Pf/(i) $\Rightarrow$ SINCE A, C BOTH ON $\mathcal{C}$
BOTH $\angle BAD$ AND $\angle BCD$ SUBTEND SAME ARC $\widehat{BD}$, SAME MEASURE.



$\Leftarrow$/ ASSUME $\measuredangle BAD = \measuredangle BCD$
BUT ABCD NOT ON SAME CIRC.
LET $\mathcal{C}$ BE CIRCLE CONTAINING A, B, D
WITH C NOT ON $\mathcal{C}$ GET CONTRADICTION.



EITHER C IS EXTERIOR TO $\mathcal{C}$ OR INTERIOR.

ASSUME C IS EXTERIOR TO $\mathcal{C}$.

LET E BE THE POINT ON $BC \cap \mathcal{C}$ WITH $B * E * C$

E IS ON $\mathcal{C}$, SUBTENDS $\widehat{BD}$.

$\measuredangle BAD = \measuredangle BED$

BUT $\measuredangle BED$ IS EXTERIOR TO ΔDEC
SO $\measuredangle BED > \measuredangle BCD = \measuredangle BAD$ BY HYP.

IF C INTERIOR, RESULT IS SAME JUST $\measuredangle BED < \measuredangle BCD$.

NEXT TIME DEAL WITH A, C IN OPP HALFS.