Circles

Monday, November 9, 2020 6:09 PM

DEF: THE CIRCLE $\mathcal{C}$ WITH CENTER O RADIUS r = $\{P : |OP| = r\}$ $\mathcal{C}_{O,r}$

THE CLOSED DISK $\overline{\mathcal{D}} = \{P : |OP| \le r\}$

OPEN DISK = $\{P : |OP| < r\}$

IN MATH
CIRCLE ≠ DISK
OFTEN CONFUSION BETWEEN THE TWO.

IF ρ ANY ROTATION WITH CENTER O, $\rho(\mathcal{C}) = \mathcal{C}$, CIRCLE IS ROTATIONALLY SYMMETRIC

DEFINED ROTATION IN TERMS OF CIRCLE: ROTATE P ABOUT O BY Θ DEGREES:

$\rho_{O,\Theta} \rightarrow$

THM: IF ℓ IS ANY LINE CONTAINING O THE $\Lambda_\ell(\mathcal{C}) = \mathcal{C}$

REFLECTION THRU ℓ

PF/ MUST SHOW IF $P \in \mathcal{C}$, THEN $\Lambda(P) \in \mathcal{C}$

$\Lambda(\mathcal{C}) \in \mathcal{C}$

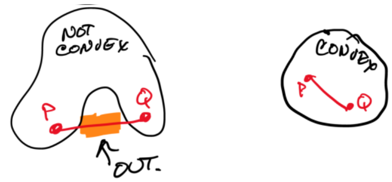
AND ALSO $\mathcal{C} \subset \Lambda(\mathcal{C})$ - -

CONVERSE: IF $\mathcal{C}$ IS A CIRCLE $\Lambda_\ell(\mathcal{C}) = \mathcal{C}$ THEN THE CENTER OF $\mathcal{C}$ IS ON ℓ.

THM: THE CLOSED DISK $\overline{\mathcal{D}}$ IS CONVEX

DO THIS IN HW#2 PROB 3. (ASSUME A FACT ABOUT TANGENT.)

$\overline{\mathcal{D}}$ IS CONVEX MEANS ANY TWO POINT IS $\overline{\mathcal{D}}$, THEY CAN BE CONNECTED BY SEGMENT LYING INSIDE $\overline{\mathcal{D}}$ WITHOUT LEAVING.



USE "LONGER SIDE FACE BIGGER ANGLE" IN A Δ THEOREM.

CASE I: $\measuredangle Q \ge 90^\circ$

$|OQ| < |OP| < r$

SINCE $\measuredangle Q > \measuredangle P$

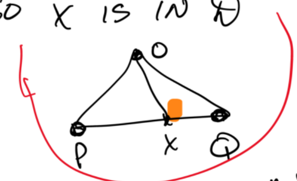
TAKE X ON $\overline{PQ}$:

$\measuredangle OXP > \measuredangle OQX$ CUZ EXTERIOR

SO $|OQ| < |OX| < |OP| < r$

SO X IS IN $\overline{\mathcal{D}}$

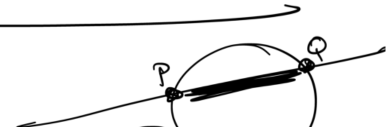
CASE 2



ONE OF THE ANGLES AT X WILL BE $\ge 90^\circ$

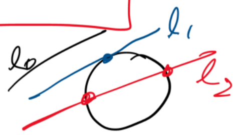
SAY IT'S $\angle OXQ$. THEN $|OX| < |OQ| \le r$

COR: IF P, Q DISTINCT ON $\mathcal{C}$ THEN $\overleftrightarrow{PQ} \cap \overline{\mathcal{D}} = \overline{PQ}$

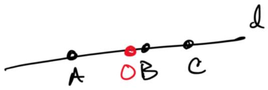


THM: A circle $\mathcal{C}$ and a line ℓ ~~meet~~ in at most 2 points

Pf/ Spoze there are (at least) 3 points on $\ell \cap \mathcal{C}$. Call them A, B, C. $A * B * C$ on ℓ



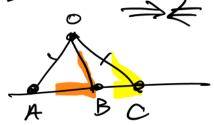
CASE 1: $O \in \ell$



since $A, B, C \in \mathcal{C}$
$|OA| = |OB| = |OC|$
since $|OA| = |OC|$, either $A = C$ or O is midpoint of $\overline{AC}$

But then since $|OB| = |OA| = |OC|$
either $B = A$ or $B = C$ both trouble $\Rightarrow\Leftarrow$

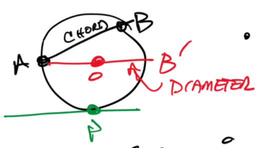
CASE 2: $O \notin \ell$



By ext. angle thm
$\measuredangle OBA > \measuredangle OCA$

But $\triangle AOB$ is isosc, $\triangle BOC$ is isosc. $\Big\} \Rightarrow \measuredangle OBA = \measuredangle OCA$
$\Rightarrow\Leftarrow$

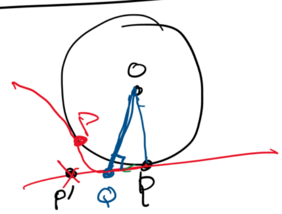
DEF: If $\ell \cap \mathcal{C} = \{\emptyset\}$ then ℓ is disjoint from $\mathcal{C}$
If $\ell \cap \mathcal{C} = \{A, B\}$:
- $\overline{AB'}$ is a diameter if $O \in \overline{AB'}$
- else $\overline{AB}$ is chord of $\mathcal{C}$; $\overleftrightarrow{AB}$ is a secant.



$\ell \cap \mathcal{C} = \{P\}$: ℓ is tangent to $\mathcal{C}$ at P.

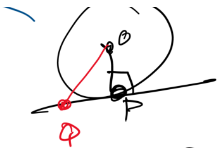
THM: Let $P \in \mathcal{C}$. A line containing P is tangent to $\mathcal{C}$ $\Leftrightarrow$ $\ell \perp \overline{OP}$

$\Rightarrow$/ Spoze ℓ tangent. Want to show $\ell \perp \overline{OP}$.
If not $\perp$ at P, there is Q so that $\overline{OQ} \perp \ell$.
Reflect thru $\overleftrightarrow{OQ}$ Λ
since $\overline{OQ} \perp \ell$, $\Lambda(\ell) = \ell$



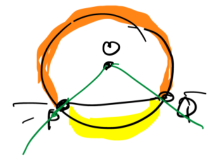
Then $\Lambda(P) = P'$ is on ℓ,
and P, P' are in opp. halfs wrt $\overleftrightarrow{OQ}$ so $P \neq P'$
$P' \in \mathcal{C}$ (since $\Lambda(\mathcal{C}) = \mathcal{C}$ since $\overleftrightarrow{OQ}$ contains O)
so $\ell \cap \mathcal{C} = \{P, P'\}$ $\Rightarrow\Leftarrow$

$\Leftarrow$/ Spoze $\ell \perp P$
Let $Q \in \ell$ with $Q \neq P$



$\triangle OPQ$ is right.
By Pythagorean thm $|PQ|^2 + |OP|^2 = |OQ|^2$
so $|OQ| > r$, so Q is exterior to $\mathcal{C}$ ($r = |OP| < |OQ|$)

If $\overline{PQ}$ is a chord of $\mathcal{C}$
Let $\mathcal{H}^+, \mathcal{H}^-$ half planes of $\overleftrightarrow{PQ}$. $(\mathcal{H}^+ \cap \mathcal{C})$ and $(\mathcal{H}^- \cap \mathcal{C})$ are two arcs of $\mathcal{C}$, both are denoted $\widehat{PQ}$



Also can define $\widehat{PQ}$ as $\angle POQ \cap \mathcal{C}$

If
- convex $\measuredangle POQ < 180°$: $\widehat{PQ}$ is a minor arc
- nonconvex $> 180°$: $\widehat{PQ}$ is a major arc
- $\measuredangle POQ = 180°$: both arcs are semicircles

$\widehat{PAQ}$ to remove ambiguity

THM: Let $\overline{PQ}$ be a chord of $\mathcal{C}$, and $A \neq P$, $A \neq Q$, $A \in \mathcal{C}$
$\measuredangle PAQ = 90° \Leftrightarrow \overline{PQ}$ is a diameter

Pf/ $\Rightarrow$ Spoze $\overline{PQ}$ is a diameter, show $\measuredangle PAQ = 90°$

$|OP| = |OA| = |OQ|$ since P, Q, A on $\mathcal{C}$
$\measuredangle OPA = \measuredangle OAP$ $\quad$ $\measuredangle OAQ = \measuredangle OQA$
Angle $\angle OAP$ and $\angle OAQ$ are adjacent

so $\measuredangle OPA + (\measuredangle OAP + \measuredangle OAQ) + \measuredangle OQA = 180°$

$2\measuredangle OPA + 2\measuredangle OAQ = 180°$
$\measuredangle OPA + \measuredangle OAQ = 90°$
$\Rightarrow \measuredangle PAQ = 90°$