Ceva

Wednesday, November 4, 2020 5:56 PM

CEVA'S ~~THEOREM~~ (CEVA. ~1678)

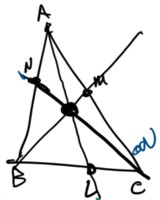
TRIANGLE ABC

L ON $\overline{BC}$, M ON $\overline{AC}$, N ON $\overline{AB}$

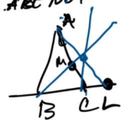
THEN. THE PRODUCT

$$\frac{|AN|}{|NB|}\cdot\frac{|BL|}{|LC|}\cdot\frac{|CM|}{|MA|}=1$$

IF AND ONLY IF $\overleftrightarrow{AL}$, $\overleftrightarrow{BM}$, $\overleftrightarrow{CN}$ ARE CONCURRENT

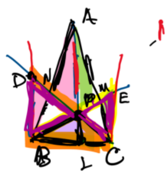


IN FACT, IT STILL WORKS IF L, M, N ARE NOT ON SIDES, BUT JUST ON LINES (BUT PROOF IS HARDER)



PROOF

⇐ ASSUME LINES ARE CONCURRENT & SHOW PRODUCT IS 1.



CREATE $\overleftrightarrow{BD}$ // TO $\overleftrightarrow{AL}$ WITH D ON $\overleftrightarrow{CN}$

AND $\overleftrightarrow{CE}$ // $\overleftrightarrow{AL}$ WITH E ON $\overleftrightarrow{BM}$

(WHY DO THEY HAVE TO MEET?)

P IS PT OF CONCURRENCE

SINCE $\overleftrightarrow{AP}$ // $\overleftrightarrow{DB}$,

$\triangle DBN \sim \triangle PAN$

(VERT ANGLES, ALT. INT)

$$\frac{|AN|}{|NB|}=\frac{|AP|}{|BD|}$$

ALSO $\overleftrightarrow{PL}$ // $\overleftrightarrow{EC}$

$\triangle APM \sim \triangle CME$

$$\frac{|CM|}{|AM|}=\frac{|CE|}{|AP|}$$

SINCE $\overleftrightarrow{PL}$ // $\overleftrightarrow{EC}$ ⇒ $\frac{|BL|}{|LC|}=\frac{|BP|}{|PE|}$

MULTIPLY ='s.

$$\frac{|AN|}{|NB|}\cdot\frac{|BL|}{|LC|}\cdot\frac{|CM|}{|MA|}=\frac{\cancel{|AP|}}{|BD|}\cdot\frac{|BP|}{|PE|}\cdot\frac{|CE|}{\cancel{|AP|}}$$

$$=\frac{|BP|}{|PE|}\cdot\frac{|CE|}{|DB|} \qquad \frac{1}{|BP|/|PE|}$$

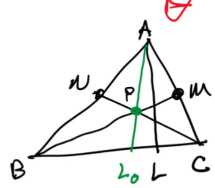
ALSO: $\triangle DBP \sim \triangle CEP$

$$\frac{|BP|}{|EP|}=\frac{|DB|}{|CE|}$$

1.

CONVERSE:

LET $\frac{|AN|}{|NB|}\cdot\frac{|BL|}{|LC|}\cdot\frac{|CM|}{|MA|}=1$



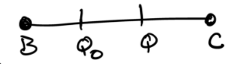
SPOSE NOT CONCURRENT.

SHOW CONTRADICTION.

LET P BE $\overleftrightarrow{CN} \cap \overleftrightarrow{BM}$, AND LET L_0 BE INTERSECTION OF $\overleftrightarrow{AP}$ WITH $\overleftrightarrow{BC}$

THEN SHOW $L=L_0$.

LEMMA:



(IE Q, Q_0 ARE ON $\overline{BC}$ BETWEEN B & C)

IF $\frac{|BQ|}{|QC|}=\frac{|BQ_0|}{|Q_0C|}$ THEN $Q=Q_0$.

IE. $|BQ|\cdot|Q_0C| = |BQ_0|\,|QC|$

SPOSE $Q \neq Q_0$ (WLOG $B*Q_0*Q$) WITHOUT LOSS OF GENERALITY.

THEN $|BQ|>|BQ_0|$, $|Q_0C|>|QC|$

$|BQ|\,|Q_0C| > |BQ_0|\,|QC|$ ⇒⇐

USE ⇐

$$\frac{|AN|}{|NB|}\cdot\frac{|BL_0|}{|L_0C|}\cdot\frac{|CM|}{|MA|}=1=\frac{|AN|}{|NB|}\frac{|BL|}{|LC|}\frac{|CM|}{|MA|}$$

HYP.

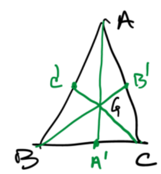
SINCE LINES $\overleftrightarrow{AL_0}$, $\overleftrightarrow{CN}$ & $\overleftrightarrow{BM}$ ARE CONCURRENT

APPLY LEMMA $L_0=L$. ☒

THIS ACTUALLY SHOWS {MEDIANS, ALTITUDES, ANGLE BISECTORS} ARE CONCURRENT

- I DON'T RECOMMEND THAT YOU USE CEVA AND SKIP THE DIRECT WAY.

MEDIANS ARE CONCURRENT AT G



(DRAWBACK: DONT GET G IS 2/3 OF THE WAY FROM VERTEX TO SIDE)

LET $A'=\text{MID}(\overline{BC})$ $C'=\text{MID}(\overline{AB})$

$B'=\text{MID}(\overline{AC})$

SINCE MIDPOINTS,

$|AC'|=\frac{1}{2}|AB|=|C'B|$ ETC.

IE $\frac{|AC'|}{|C'B|}=1=\frac{|BA'|}{|A'C|}=\frac{|CB'|}{|B'A|}$ SO CONCURRENT AT G

ALTITUDES ARE CONCURRENT

(PROOF WORKS IF ABC IS OBTUSE BUT BE CAREFUL

IF D*B*C, NEED THE EXTENDED CEVA

LOTS OF SIMILAR TRIANGLES.

$\triangle AFC \sim \triangle AEB$. SO $\frac{|AF|}{|AE|}=\frac{|AC|}{|AB|}$

SHARE A, ∟

$\triangle ABD \sim \triangle CBF$ SO $\frac{|BD|}{|BF|}=\frac{|BA|}{|CB|}$

$\triangle ADC \sim \triangle BEC$ SO $\frac{|CE|}{|CD|}=\frac{|CB|}{|AC|}$

WANT TO BE 1

$$\frac{|AF|}{|FB|}\cdot\frac{|BD|}{|DC|}\cdot\frac{|CE|}{|EA|}=\frac{|AF|}{|AE|}\cdot\frac{|BD|}{|BF|}\cdot\frac{|CE|}{|DC|}$$

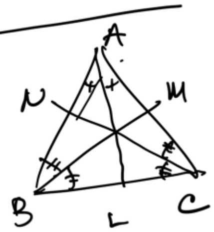
$$=\frac{\cancel{|AC|}}{\cancel{|AB|}}\cdot\frac{\cancel{|BA|}}{\cancel{|BC|}}\cdot\frac{\cancel{|CB|}}{\cancel{|AC|}}=1$$

☺

FOR ANGLE BISECTORS, NEED A THM

THM IN $\triangle ABC$, ANGLE BISECTOR OF $\angle A$ INTERSECTS BC AT L

$$\frac{|BL|}{|LC|} = \frac{|AB|}{|AC|}$$



ONCE WE PROVE THIS, THEN BISECTORS CONCURRENT IS EASY:

$$\frac{|AN|}{|NB|}\cdot\frac{|BL|}{|LC|}\cdot\frac{|CM|}{|MA|} = \frac{|\cancel{AC}|}{|\cancel{BC}|}\cdot\frac{|\cancel{AB}|}{|\cancel{AC}|}\cdot\frac{|\cancel{BC}|}{|\cancel{AB}|}$$

$= 1$ SO CONCURRENT.

SO NOW TO PROVE THM.

TRICK IS TO TURN IT INTO SOMETHING I CAN USE FTS.

EXTEND $\overrightarrow{CA}$ TO D WITH $|AD| = |BA|$

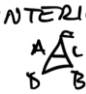
SO $\measuredangle BDA = \measuredangle DBA$

$\measuredangle BAC$ IS ETERIOR TO $\triangle DBA$

$\measuredangle BAC = \measuredangle BDA + \measuredangle DBA = 2\measuredangle DBA = 2\measuredangle BAL$

SO $\measuredangle DBA = \measuredangle BAL$.

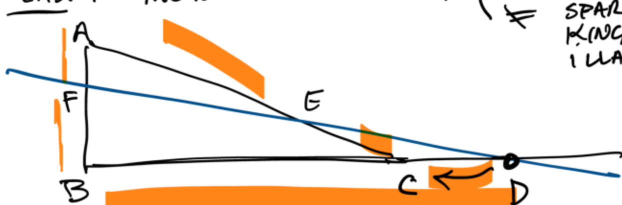
SO BY ALT. INTERIOR, $\overleftrightarrow{DB} \parallel \overleftrightarrow{AL}$

APPLY FTS TO  TO GET

$$\frac{|BL|}{|LC|} = \frac{|DA|}{|AC|} = \frac{|AB|}{|AC|}$$

SO DONE.

LAST: MENELAUS' THM. (~100 AD ASTRONOMER $\neq$ SPARTAN KING FROM ILIAD)



$$\frac{|AF|}{|FB|}\cdot\frac{|BD|}{|DC|}\cdot\frac{|CE|}{|EA|} = 1$$

(CAN WRITE AS SIGNED DISTANCES & PRODUCT IS -1)