Comments about the exam

B IS BETWEEN A & C (ON LINE $\overleftrightarrow{AB}$)

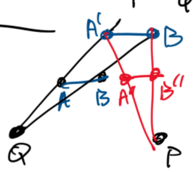
TWO OPTIONS:

- B IS ON THE INTERSECTION $\overrightarrow{AC}$ AND $\overrightarrow{CA}$ BUT $B \neq A$, $B \neq C$.
- IF WE ASSOCIATE $\overleftrightarrow{AB}$ WITH A NUMBERLINE VIA $\#(\cdot)$, THEN $\#(A) < \#(B) < \#(C)$ OR $\#(A) > \#(B) > \#(C)$

#3 MOST PEOPLE DID THIS:

WANT TO SHOW THAT DILATIONS Δ_P w/ r, Δ_Q w/ $1/r$

THEN $\Delta_P \circ \Delta_Q$ IS A TRANSLATION.



& SHOWED $|AB| = |A''B''|$

ALL THIS DOES IS SHOW THAT F IS AN ISOMETRY.

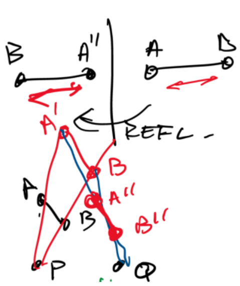
$F = \Delta_P \circ \Delta_Q$.

F IS A TRANSLATION OR ROTATION OR REFLECTION OR COMPOSITION OF THOSE.

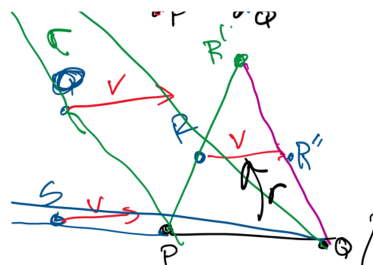
ALSO $\overleftrightarrow{AB} \parallel \overleftrightarrow{A''B''}$ (OR A, B, A'', B'' COLINEAR)

THEN:

$|AB| = |A''B''|$ & $\overleftrightarrow{AB} \parallel \overleftrightarrow{A''B''}$



TRANS, REFL. OR ROTATION BY 180°



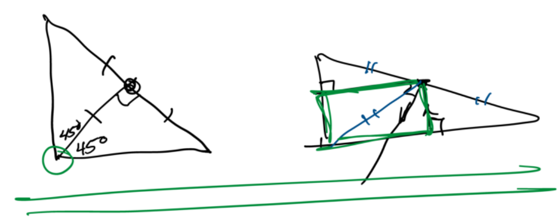
GIVEN P, Q Δ_r, $\Delta_{1/r}$

MUST SHOW THAT THERE IS A VECTOR V SO THAT THE IMAGE OF ANY POINT P IS $P + \vec{V}$

$$\frac{|R'R|}{|R'P|} = \frac{|R'R''|}{|R'Q|}$$

$\overleftrightarrow{RR''} \parallel \overleftrightarrow{PQ}$ & LENGTH V IS CONSTANT.

TRANSLATION IS JUST ALONG VECTOR $\parallel$ TO $\overleftrightarrow{PQ}$ BY $|PQ|/r$



The Euler Line

RECALL:

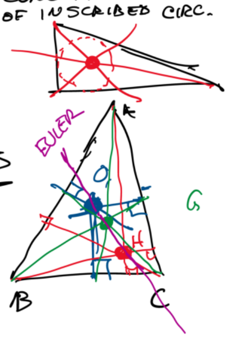
INCENTER = ANGLE BISECTORS AT VERTICES ARE CONCURRENT
= CENTER OF INSCRIBED CIRC.

O = CIRCUMCENTER
= CONC. OF ⊥ BISECTORS OF SIDES

H = ORTHOCENTER
= CONCURRENCY OF ALTITUDES

G = CENTROID = CONCURRENCY OF MEDIANS.

DIST(G, VERTEX) = 2 DIST(G, OPP SIDE)



THM: (EULER LINE)

(i) IF $H=G=O$, THEN ΔABC IS EQUILATERAL

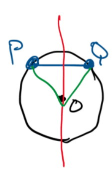
(ii) IF $G \neq O$, THEN H LIES ON $\overleftrightarrow{GO}$ WITH $H*G*O$ AND $|HG| = 2|GO|$

(INCENTER IS ON EULER ⟺ ABC IS ISOSC.)

- EULER SEGMENT IS $\overline{HO}$
- CENTER OF THE 9-POINT CIRCLE IS ON EULER LINE.

LOTS OF PROOFS OF THIS.

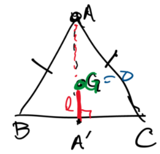
LEMMA LET $\overline{PQ}$ BE A CHORD OF A CIRCLE WITH CENTER O. THEN THE ⊥ BISECTOR OF $\overline{PQ}$ CONTAINS O.



Pf/ $\overline{OP}$ AND $\overline{OQ}$ ARE RADII
SO $|OP| = |OQ|$ SO ΔOPQ IS ISOSC. AND SO ⊥ BISECTOR FROM O = ALTITUDE FROM O.
= MEDIAN.

Pf OF EULER LINE (i)

CASE WHERE $G=O$. SHOW Δ IS EQUILATERAL.



G = CENTROID O = CIRCUMCENTER.

LET $A' = \text{MID}(B,C)$

OBSERVE G IS NOT ON ANY SIDE

SINCE $|AG| = 2|GA'|$

SO IF G ON BC, $|AG| = |GA'| = 0$. ⇒⇐

LET $\overleftrightarrow{A'G} = \ell$ = PERP BISECTOR OF $\overline{BC}$

ALSO $\overline{AA'}$ = MEDIAN FROM A, SO G ON $\overleftrightarrow{AA'}$

$A*G*A'$ SO $\overleftrightarrow{AA'} = \overleftrightarrow{AG} = \overleftrightarrow{GA'}$

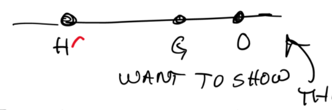
IN PARTICULAR MEDIAN = ⊥ BISECTOR OF BASE.

SO ΔABC IS ISOSCELES. $|AB| = |BC|$

NOW DO AGAIN FROM C TO SEE THAT $|CB| = |CA|$

SO EQUILATERAL.

PART (ii)



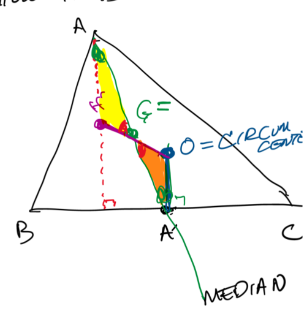
HAVE TO SHOW THAT ORTHOCENTER H IS ON $\overleftrightarrow{GO}$ AND $|HG| = 2|GO|$ AND $H*G*O$

TRICK IS TO JUST DEFINE SOME H' WITH THESE PROPERTIES & SHOW IT IS ORTHOCENTER.

CONSTRUCT G & O

PUT H' ON $\overleftrightarrow{GO}$ WITH $|H'G| = 2|GO|$

ALL I NEED TO DO IS SHOW THAT H' IS ON ANY ALTITUDE.



G IS ON MEDIAN $\overline{AA'}$

LOOK AT $\Delta GAH'$

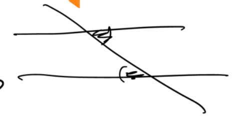
FROM BEFORE $|AG| = 2|A'G|$

BY THE WAY WE CONSTRUCTED H'
$|H'G| = 2|OG|$

$\angle AGH' = \angle A'GO$ SO TWO SIDES IN PROPORTION & SAME ANGLE,

SO $\Delta AH'G \sim \Delta A'OG$

SO $\angle A = \angle A'$



SO BY ALT. INT. ANGLES
$\overleftrightarrow{OA'} \parallel \overleftrightarrow{AH'}$

SO BOTH ARE ⊥ TO $\overline{BC}$

SO $\overleftrightarrow{AH'}$ IS THE ALTITUDE FROM A

SO H' IS ON THE ALTITUDE.

PLAY SAME GAME FROM OTHER VERTEX TO SEE $H' = H$ = ORTHOCENTER.