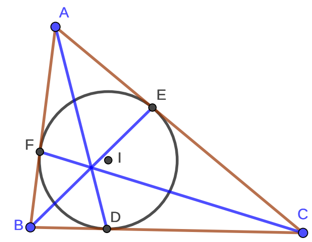
MAT515 Homework 9

Due Wednesday, November 18

1. Given triangle $\triangle ABC$, recall that the incircle is the unique circle that is tangent to the three lines $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, and $\overleftrightarrow{AC}$. Its center is the **incenter**, which is the interscetion of the three angle bisectors of the triangle.



Let the points of tangency of the incircle be D on $\overline{BC}$, E on $\overline{AC}$ and F on $\overline{AB}$. Show that the segments $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are concurrent. The point of concurrency is called the **Gergonne point** of $\triangle ABC$.

2. Prove that the incircle of the right triangle with side lengths 3, 4, and 5 (usually called the "3-4-5 triangle") has radius 1.

A generalization of this result is that in a right triangle with legs of length a and b and hypotenuse of length c, the radius of the incircle is $\frac{1}{2}(a+b-c)$.

3. Let two circles $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect at distinct points M and N.

(a) Prove that the line joining the center of $\mathcal{C}_1$ to the center of $\mathcal{C}_2$ is perpendicular to $\overleftrightarrow{MN}$.

(b) Let the line ℓ be tangent to both $\mathcal{C}_1$ and $\mathcal{C}_2$. Prove that the circles have equal radii if and only if $\overleftrightarrow{MN}$ is perpendicular to ℓ.

4. Let $\triangle ABC$ be an acute triangle with altitudes $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$. Triangle $\triangle DEF$ is called the **orthic triangle** of $\triangle ABC$. Prove that the orthocenter of $\triangle ABC$ is the incenter of its orthic triangle $\triangle DEF$.

Among other properties, the orthic triangle has the smallest perimeter among all triangles with vertices on the sides of $\triangle ABC$. This solves Fagano's problem, first stated in 1775.