MAT515 Homework 2
Due Wednesday, September 16

1. Prove that every angle has exactly one angle bisector.
Hint: use axiom (L6). Make sure your proof also works for a full angle and a zero angle.

Then explain why this shows that Lemma 4.9 in the text holds.

Lemma 4.9. *Let ℓ be a line and O be a point on ℓ. Then there is one and only line one passing through O which is perpendicular to ℓ.*

2. Let $\mathcal{C}$ be a circle.

(a) Prove that the exterior of $\mathcal{C}$ has $\mathcal{C}$ as its boundary.

(b) Prove that the exterior of $\mathcal{C}$ cannot be convex.

3. Let P be a point on a circle $\mathcal{C}$. Then a line ℓ_P is defined to be **tangent to $\mathcal{C}$ at P** if it intersects $\mathcal{C}$ exactly at P, that is, $\ell_P \cap \mathcal{C} = \{P\}$.

Assume that every point P of a circle $\mathcal{C}$ has a unique tangent line through it (we will prove this later) and that the circle $\mathcal{C}$ lies entirely within one of the closed half-planes of *every* such tangent.

Use the above to prove that any disk (whether open or closed) is convex.

4. Assume that for any subset $\mathcal{S}$ of the plane, if $X \in \mathcal{S}$ and $Y \notin \mathcal{S}$, then $\overline{XY}$ intersects the boundary of $\mathcal{S}$. (This assumption is nontrivial, and requires the Least Upper Bound axiom for $\mathbb{R}$).

Using this assumption, prove that if $\mathcal{S}$ is any closed, bounded set in the plane and which has a circle $\mathcal{C}$ as its boundary[†], then $\mathcal{S}$ is actually the closed disk with the same center and radius as $\mathcal{C}$.

5. A transformation F of the plane is an **isometry** if it preserves distances. That is, for all points P and Q, we have $\operatorname{dist}(P,Q) = \operatorname{dist}(F(P),F(Q))$.

(a) Prove that the composition of two isometries is an isometry.

(b) Prove that the composition of two surjections is a surjection.

(c) Prove that the composition of two injections is an injection.

6. If F is a transformation such that $F \circ F = F$, then F is called **idempotent**.

(a) Show that both the identity transformation I and the constant transformation are idempotent.

(b) Give an example of a transformation of the plane F which is idempotent but is neither the identity nor a constant transformation.

(c) Let F be an injective transformation of the plane which is idempotent. Prove that F is the identity transformation.

(d) Let F be a surjective idempotent transformation of the plane. Prove that F is the identity transformation.

[†]This problem is wrong as originally written. It needs to include "and $\mathcal{S} \neq \mathcal{C}$", since $\mathcal{C}$ is a closed bounded set with $\mathcal{C}$ as its boundary. Alternatively, it could say $\mathcal{S}$ is a closed, bounded set *with nonempty interior*, except we never defined the interior of a set. Here is the definition: A point X is **in the interior of $\mathcal{S}$** if there is an $\varepsilon > 0$ so that every point of the disk of radius ε centered at X is also in $\mathcal{S}$.