MAT515 Final Exam

Due Wednesday, December 9

This is a **open book/notes/internet, take-home** exam. That means you can use outside sources while doing these questions, **but you must credit any sources you use** other than classnotes or the textbook. Failure to do so constitutes a violation of academic integrity.

Some of the problems are easier than others, but they all count for the same amount of your grade.

1. Do as many of the constructions in Euclid: the Game as you can. (There are 25 levels — you don't have to do them all, but do what you can.) Show what you have done by turning in a screenshot like the one at right. Here I have done levels 1–20. **Warning**: it doesn't always remember your progress, so take some screenshots as you go to be safe. If you want to go to a specific level, you can use a URL like `http://www.euclidthegame.com/Level18`.

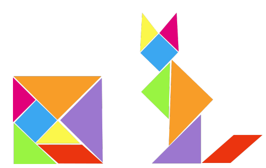
2. State and prove the theorem of Menelaus. You can find several slightly different proofs of this on the internet, or devise your own. Write this out completely, in your own words, together with an example or two. Treat this as though you are teaching this theorem to a geometry class. (Don't forget to credit your sources.)

3. We haven't disucssed area in this class, so you get to do that on this final. Area in general is a somewhat tricky subject to get fully correct, but we will stick to the easy case of area of polygons; in fact, just triangles.

A1: The area of a rectangle of height h and width w is $h \times w$ square units.
(This follows readily from the more basic idea "the area of a 1×1 square is 1 square unit" as long as h and w are rational numbers, but you can use it in this form.)

A2: If plane figure $\mathcal{P} = P_1 \cup P_2$ where P_1 and P_2 are polygons with disjoint interiors, then the area of $\mathcal{P}$ is equal to the sum of the areas of P_1 and P_2.

A3: Suppose plane figures $\mathcal{P}$ and $\mathcal{Q}$ can be subdivided into polygons $P_1, P_2, \ldots, P_n$ and $Q_1, Q_2, \ldots, Q_n$ such that $P_i \cong Q_i$ for $1 \le i \le n$, $\mathcal{P} = \bigcup P_i$, $\mathcal{Q} = \bigcup Q_i$, and P_i and P_j have disjoint interiors (and similarly for $\mathcal{Q}$). In this case, $\mathcal{P}$ and $\mathcal{Q}$ are said to be **scissors congruent**. For example, in the figure at right, the square is scissors congruent to the "cat".



If $\mathcal{P}$ is scissors congruent to $\mathcal{Q}$, then $\mathcal{P}$ and $\mathcal{Q}$ have the same area.

Using these assumptions, *carefully* prove that the area of a triangle is half of the length of its base times the length of the altitude to that base. Note that every triangle has three choices of base. You might find it easiest to start with right triangles.

Then discuss (but don't prove in general) how this result can be applied to find the area of arbitrary polygons.

You might find it interesting to know that while any two polygons of equal area are scissors congruent (this is the Wallace-Bolyai-Gerwien theorem, proven between independently in 1803, 1833, and 1835), the analogous question in three dimensions (whether any two polyhedra of equal volume can be cut into finitely many congruent pieces) is false. For example, a cube and a regular tetrahedron are not scissors congruent. This is Hilbert's Third Problem, posed in 1900 and solved shortly thereafter by Max Dehn.

4. There are several approaches to teaching Geometry in schools. Some schools teach it almost as a hodge-podge of seemingly unrelated concepts and formulae, but I hope we can agree that this is not actually teaching *mathematics* and is not a good approach despite its widespread adoption.

One very common mathematical approach is to teach geometry from the point of view of constructions (this is essentially the approach of Euclid from 2300 years ago), although this approach is now fairly rare in school. Since the middle of the 20th century, another approach came into favor, building up geometry from a set of axioms (one common set of axioms is developed explicitly for this purpose is the School Mathematics Study Group (SMSG) set of axioms, but there are many variations). This is often coupled with a very formalized method of proof (two-column proofs), although this method of presenting proofs is not strictly necessary for this approach.

The approach we have taken in this class, by defining congruence and similarity through isometries and and dialations, is somewhat unusual although it ultimately leads to completely equivalent theorems and results.

Discuss the advantages and disadvantages of the approach taken in this class, comparing and contrasting with other approaches. Keep in mind that you may find yourself teaching geometry in the near future. In your discussion, you should take into account the Common Core Standards, not only for high school but earlier grades as well.

Note that this question does not have a “correct answer”. Your grade on this question will depend on how well you present and support your argument.