MATH 515 Solutions to Midterm 2

5 pts. 1. (a) Give the definition of the **median** of a triangle.

Solution: The median of a triangle is a segment bounded by a vertex of the triangle and the midpoint of the opposite side.

5 pts. (b) For a circle $\mathcal{C}$ with radius r, define **diameter** in two ways (as a number and as a geometric object).

Solution: A diameter of $\mathcal{C}$ is a chord which contains the center. (Alternatively: the diameter is a chord of length $2r$.)
The diameter of $\mathcal{C}$ is the length of any such chord. (Alternatively, the diameter is $2r$.)

5 pts. (c) Let $\mathcal{P}$ be a convex polygon. Give the definition of what it means for a circle $\mathcal{C}$ to be **inscribed** in $\mathcal{P}$.

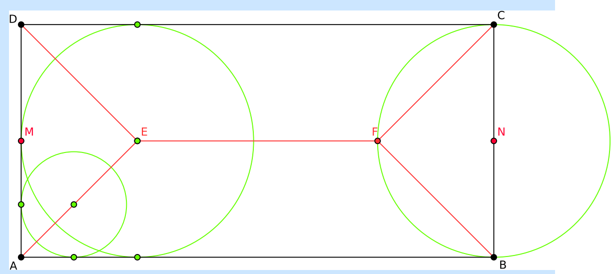
Solution: The circle $\mathcal{C}$ is inscribed in $\mathcal{P}$ if each side of $\mathcal{P}$ is tangent to $\mathcal{C}$.

15 pts. 2. Let $\mathcal{R}$ be a rectangle whose width is greater than its height. What is the geometric locus of all points O which are centers of circles $\mathcal{C}$, where each circle $\mathcal{C}$ is tangent to $\mathcal{R}$ in at least two points?

I meant to add that the circles intersect $\mathcal{R}$ only in points of tangency. If you allow the circles to intersect $\mathcal{R}$ in additional points (as well as the two or more points of tangency), please make this clear.

Solution:

Let E be the intersection of the bisector of angle A with the bisector of angle D, and let F be the intersection of the bisectors of angles B and C. Further, let M be the midpoint of $\overline{AD}$ and N be the midpoint of $\overline{BC}$.
Then the locus consists of the five segments $\overline{AE}$, $\overline{DE}$, $\overline{EF}$, $\overline{BF}$, and $\overline{CF}$, together with the points M and N, but not the points A, B, C, or D.



Here is the GeoGebra file of the above.

There are a few cases to consider:

- **$O = E$ (resp. $O = F$) :** In this case, the circle centered at O will be tangent to both $\overline{AB}$ and $\overline{CD}$, as well as $\overline{AD}$ (resp. $\overline{BC}$). This is because it lies on the bisector of the lower angle (either A or B) and hence is equidistant from both the side and the bottom edge. As such, it will be tangent to each. Similarly, it is equidistant from the side and the top. This means there will be three points of tangency: on the top, the bottom, and one of the sides.

- O **on** $\overline{EF}$ **:** Observe that since $\mathcal{R}$ is a rectangle, $\overline{EF}$ will be parallel to both the top and the bottom of $\mathcal{R}$. Since E is equidistant from the top and bottom of R, all points of $\overline{EF}$ will also be. This means that any circle centered on $\overline{EF}$ will be tangent to both the top and the bottom of $\mathcal{R}$.
- O **on** $\overline{AE}$ **:** In this case, O is equidistant from the bottom $\overline{AB}$ and the side $\overline{AD}$, so the circle is tangent to these two sides.
- O **on** $\overline{DE}$**,** $\overline{BF}$**, or** $\overline{CF}$ **:** a similar argument applies.
- $O = M$ **(resp** $O = N$**) :** If O is the midpoint of $\overline{AD}$, then $\overline{AD}$ will be a diameter of the circle, and the circle will be tangent at the top and the bottom. A similar thing happens when $O = N$, using instead $\overline{BC}$ as diameter.

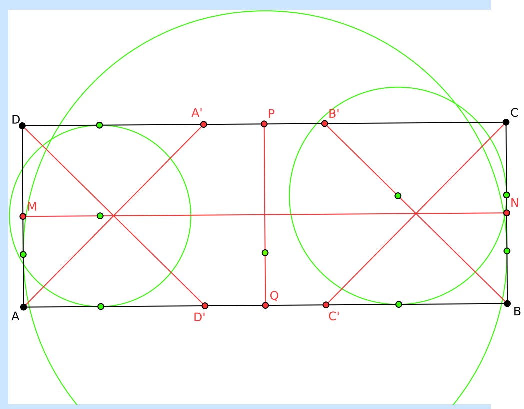
If O is not on one of the above segments, it will be closer to one side of $\mathcal{R}$ than any other, and hence either will be tangent in only one point, or will intersect the closer side in more than one point.

Now we turn to what some people thought I meant: the circles still have two or more tangencies, but are allowed to intersect $\mathcal{R}$ in other points that are not tangencies.

Let A' be the intersection of the bisector of $\angle A$ with $\overline{CD}$, B' be the intersection of the bisector of $\angle B$ with $\overline{CD}$, C' be the intersection of the bisector of $\angle C$ with $\overline{AB}$, & D' be the intersection of the bisector of $\angle D$ with $\overline{AB}$. Also let M and N be the midpoints of $\overline{AD}$ and $\overline{BC}$, and also let P and Q be the midpoints of $\overline{CD}$ and $\overline{AB}$.

Then the locus consists of the six segments $\overline{AA'}$, $\overline{BB'}$, $\overline{CC'}$, $\overline{DD'}$, $\overline{MN}$, and $\overline{PQ}$, but not the points A, B, C, or D.

To see this, note that if the center O of a circle lies on $\overline{PQ}$, it will be tangent to the sides $\overline{AD}$ and $\overline{CB}$ (as well as intersecting the top and bottom in more points.



Here is the GeoGebra file of the above.

A circle with center on $\overline{MN}$ will be tangent to $\mathcal{R}$ on the top and the bottom, and may also intersect one of the other two sides in either one or two points.

A circle with center on one of the angle bisectors such as $\overline{AA'}$ will be tangent to the sides of the angle it bisects. It may also intersect one of the other sides in one or two additional points.

Note that the vertices of the rectangle cannot be centers of the circles: such circles can have at most one tangency. For example, a circle centered at A cannot be tangent to $\overline{AB}$ or $\overline{AD}$, since these would be radii of the circle. It might be tangent to $\overline{DC}$, but because the rectangle is not square, it cannot also be tangent to $\overline{BC}$.

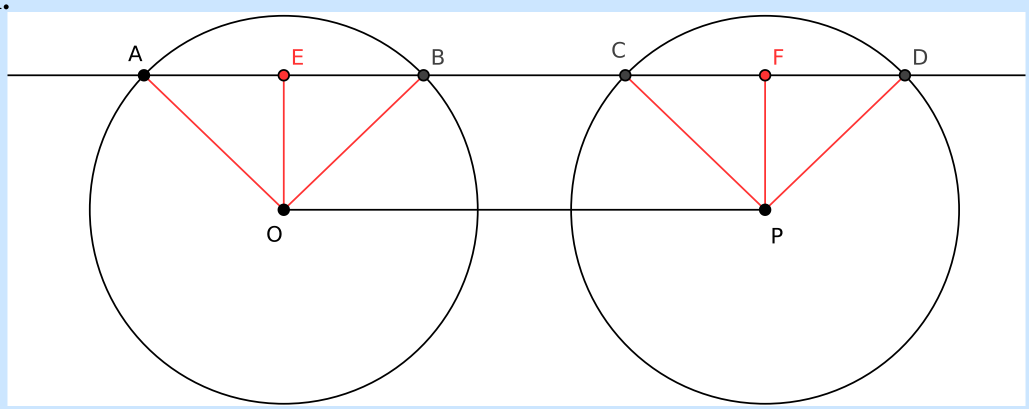
Points with centers outside of $\mathcal{R}$ cannot be tangent on the *segments* corresponding to one of the sides, since the segment from the center O to the point of tangency must be perpendicular to the corresponding side of the rectangle.

As before, a point O which is not on one of the angle bisectors cannot be equidistant from two intersecting sides of $\mathcal{R}$, and hence the circle cannot be tangent to more than one of them. If O does not lie midway between two parallel sides of $\mathcal{R}$, a circle centered at O be tangent to them simultaneously.

15 pts. 3. Let congruent circles $\mathcal{C}_1$ and $\mathcal{C}_2$ have centers O and P, respectively. Let ℓ be a secant which is parallel to $\overline{OP}$. Furthermore, suppose ℓ intersects $\mathcal{C}_1$ in points A and B, and intersects $\mathcal{C}_2$ in C and D. Finally, assume points B and C are interior to segment $\overline{AD}$.

Prove that $|AC| = |BD| = |OP|$.

Solution:



First, construct perpendiculars at O and P which meet ℓ at points E and F, respectively. Then quadrilateral $OPFE$ is a rectangle, so $|OE| = |PF|$ and $|OP| = |EF|$.

Note also that $|OA| = |OB| = |PC| = |PD|$, since these are all radii of the same (or congruent) circles. Consequently, by hypoteneuse-leg, we have

$$\triangle OAE \cong \triangle OBE \cong \triangle PCF \cong \triangle PFD.$$

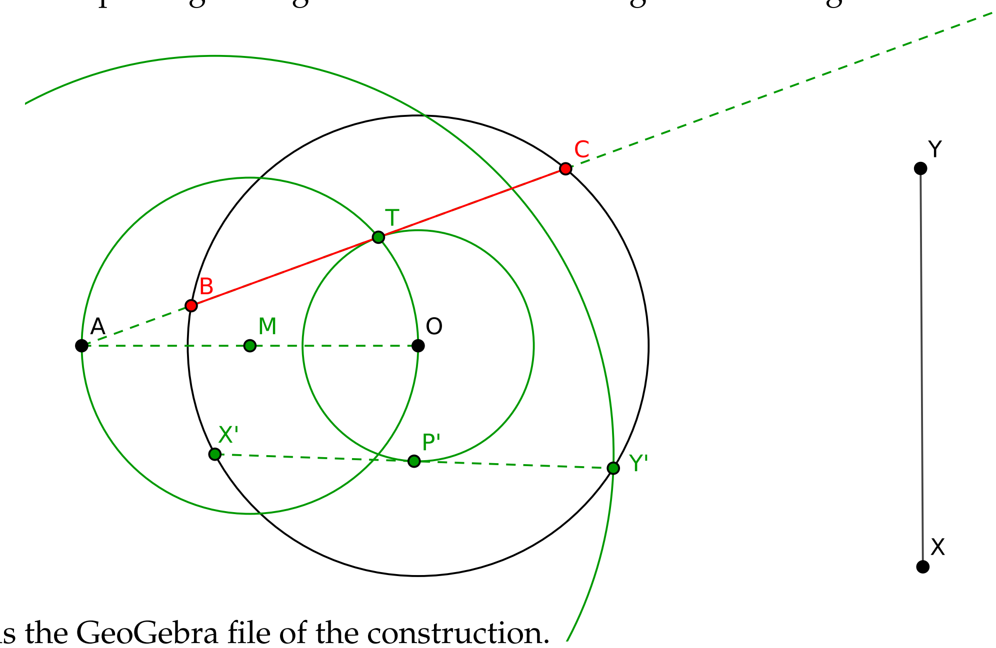
Now observe that

$$|EF| = |EF| + (|AE| - |CF|) = |AC|,$$

and since $OPFE$ is a rectangle, $|OP| = |EF| = |AC|$. Shifting the other way gives

$$|OP| = |EF| = |EF| + (|DF| - |BE|) = |BD|.$$

15 pts. 4. Given a circle $\mathcal{C}$ with center O, a point A outside the circle, and a segment $\overline{XY}$, construct a secant to $\mathcal{C}$ passing through A so that the resulting chord is congruent to $\overline{XY}$.



Here is the GeoGebra file of the construction.

Solution: The main idea is to observe that all chords of the same length are equidistant from the center. Thus, if we copy $\overline{XY}$ into $\mathcal{C}$ as a chord, we can find a smaller circle that all such chords are tangent to. Then we construct a tangent from A to this circle, and that gives the desired secant.

So, here we go:

1. Pick a point X' on the circle $\mathcal{C}$. Then set the compass to radius $|XY|$ and find another point Y' on $\mathcal{C}$ so that the chord $\overline{X'Y'}$ has length $|XY|$.
2. Find the midpoint of $\overline{X'Y'}$; call it P'.
3. Draw the circle with center O and radius $|OP'|$. Call this circle $\mathcal{D}$.
4. Now draw a tangent to circle $\mathcal{D}$ passing through A: First, find the midpoint of $\overline{AO}$ and call it M. Then draw the circle with center M and radius $|MO|$. Call the intersection of this circle with $\mathcal{D}$ the point T, and draw $\overrightarrow{AT}$.
5. We're done: let points B and C be the intersection of $\overrightarrow{AT}$ with $\mathcal{C}$. Segment $\overline{BC}$ is the chord with the desired length.

Proof: Radius $\overline{OP'}$ is the perpendicular bisector of $\overline{X'Y'}$, since radii are perpendicular to chords at their midpoint. Thus, since chords $\overline{X'Y'}$ and $\overline{BC}$ are both tangent to circle $\mathcal{D}$, they are equidistant from the center O. We know that chords at the same distance from the center of a given circle are congruent. Point A lies on $\overleftrightarrow{BC}$ by construction. Thus, $\overleftrightarrow{AT}$ is the desired secant. (A proof that $\overleftrightarrow{AT}$ is tangent is not necessary; we've done this several times in class.) This construction will work as long as $|XY|$ is not larger than the diameter of $\mathcal{C}$.