MATH 515 Solutions to Midterm 1

5 pts. 1. (a) Give the definition of a **circle** with center O and radius r.

Solution: The circle with center O and radius r is the set of all points P in the plane for which $|OP| = r$.

5 pts. (b) Give the definition of the **interior** of a triangle.

Solution: Let $\triangle ABC$ be a triangle. Then the interior of $\triangle ABC$ is the set of points P for which P is interior to $\angle ABC$ and also interior to $\angle BAC$.

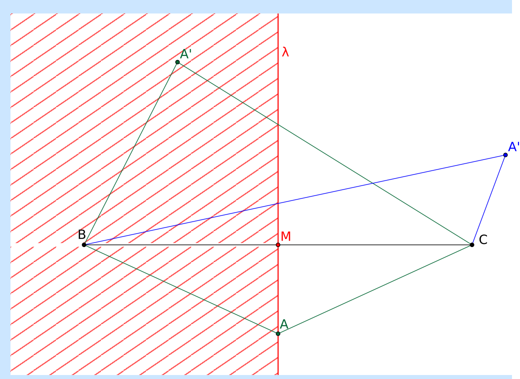
5 pts. (c) Give the definition of the **altitude** of $\triangle ABC$ with base $\overline{BC}$.

Solution: The altitude is the segment $\overline{AX}$ where X is a point on BC and $\overline{AX}$ is perpendicular to $\overline{BC}$.

15 pts. 2. Given a segment $\overline{BC}$, what is the locus of points which are the vertex A of a triangle $\triangle ABC$ with base $\overline{BC}$ and $\measuredangle B \geq \measuredangle C$?

Solution: Construct λ, the perpendicular bisector to $\overline{BC}$. Then the locus consists of the bisector λ together with all points on the same side of λ as B *except* for the points lying on $\overleftrightarrow{BC}$.

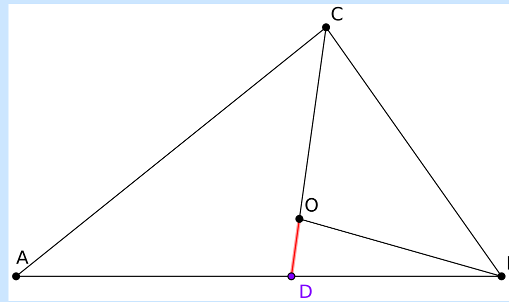
- If A lies on the line $\overleftrightarrow{BC}$, it can *not* be in the locus, because points A, B, and C are colinear and do not form a triangle.
- If A lies on the bisector λ of $\overline{BC}$ (except for the midpoint M), then the resulting triangle $\triangle ABC$ is isosceles. Thus, $\measuredangle B = \measuredangle C$, and A is in the locus (as shown by green $\triangle ABC$ in the figure).
- If A lies on the same side of λ as B (but not on $\overleftrightarrow{BC}$), then $|AC| > |AB|$, and so $\measuredangle B > \measuredangle C$. Thus A is in the locus (as shown by green $\triangle A'BC$).
- If A lies on the other side of λ from B, then $|AC| < |AB|$, and $\measuredangle B < \measuredangle C$. Thus A is *not* in the locus (as shown by blue $\triangle A''BC$).



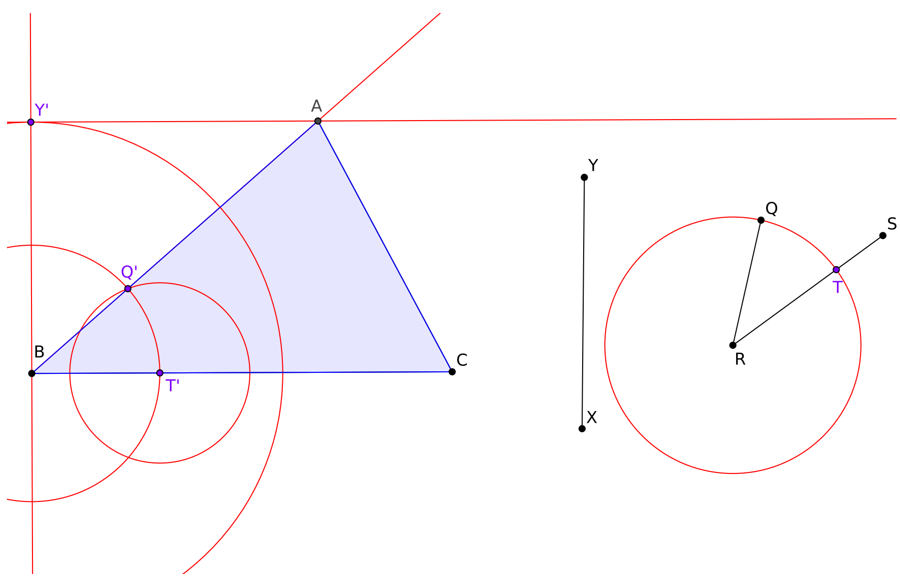
15 pts. 3. Given $\triangle ABC$ with a point O in its interior, prove that $|CO| + |BO| < |AC| + |AB|$.

Solution:
First, extend $\overline{CO}$ to meet $\overline{AB}$ at D. Then apply the triangle inequality twice (to triangles $\triangle CAD$ and $\triangle ODB$).

$$\begin{aligned} |AC| + |AB| &= |AC| + |AD| + |DB| \\ &> |CD| + |DB| \\ &= |CO| + |OD| + |DB| \\ &> |CO| + |OB|. \end{aligned}$$



15 pts. 4. Given a segments $\overline{BC}$ and $\overline{XY}$, and an angle QRS, construct a triangle with base $\overline{BC}$, altitude congruent to $\overline{XY}$ and $\angle B \cong \angle QRS$.



An animation of this construction can be found at `http://www.math.sunysb.edu/~scott/mat515.fall14/Geogebra/SegAngAlt_const.html`, or here is the GeoGebra file.

Solution:

1. Erect a perpendicular to $\overline{BC}$ at B.
2. Set the compass to $|XY|$ and locate point Y' on this perpendicular with $|BY'| = |XY|$.
3. By erecting another perpendicular, construct a line parallel to $\overline{BC}$ through Y'.
4. Draw a circle with center R and radius $|RQ|$; let T be the intersection with $\overrightarrow{RS}$.

5. Set the compass to $|RT|$ and construct the circle with center B and this radius. Denote the intersection of this circle and $\overline{BC}$ by T'.

6. Set the compass to $|TQ|$ and draw a circle with center T' and this radius. This circle intersects the circle drawn in step 4 at a point Q' on the same side of $\overline{BC}$ as Y'.

7. Extend ray $\overrightarrow{BQ'}$ to meet the parallel drawn in step 3; call this intersection point A.

8. The desired triangle is $\triangle ABC$.

Proof: The altitude of the triangle $\triangle ABC$ is congruent to $\overline{XY}$ since $|BY'| = |XY|$ and $\overleftrightarrow{AY'}$ is parallel to $\overleftrightarrow{BC}$, as constructed in step 3. Since parallel lines are the same distance apart everywhere, the distance from A to $\overline{BC}$ is the desired one.

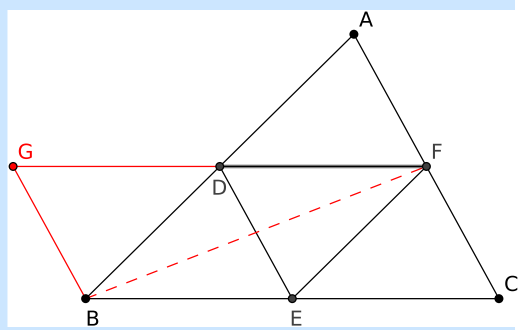
Steps 4 through 6 construct an angle $\angle CBQ'$ which is congruent to $\angle SRQ$. We have this since $\triangle T'BQ' \cong \triangle TRQ$ by the **SSS** congruence. Specifically, since $\overline{BQ'}$, $\overline{BT'}$, $\overline{RQ}$, and $\overline{RT}$ are radii of congruent circles, they are all of the same length; furthermore $|T'Q'| = |TQ|$ as constructed in step 6.

Thus, $\triangle ABC$ has the desired properties.

15 pts. 5. On $\triangle ABC$, points D, E, and F are the midpoints of sides $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$, respectively. ($\triangle DEF$ is called the *medial triangle*.) Prove that all four small triangles are congruent.

Solution: Once we show that $\overleftrightarrow{DF} \| \overleftrightarrow{BC}$ (and similarly for the other two sides of the central triangle), the rest is pretty straightforward.

First, extend $\overline{DF}$ and locate G so that $|DF| = |GD|$. Then $\angle ADF \cong \angle BDG$, since they are vertical angles. Since D is the midpoint of $\overline{BA}$, we have $|BD| = |DA|$. Finally, $|GD| = |DF|$ by construction. This means that $\triangle GDB \cong \triangle FDA$ by **SAS**.



Thus, $\angle G \cong \angle DFA$, which means $\overline{BG} \parallel \overline{AC}$ since the alternate interior angles are equal.
Also, $|BG| = |AF| = |FC|$.

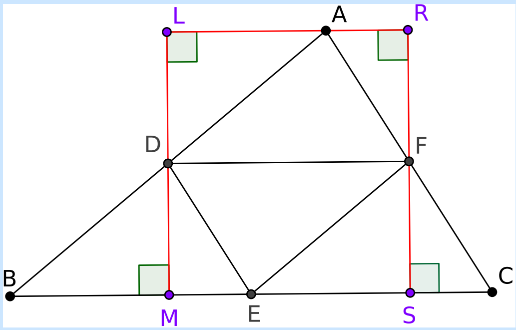
Now draw $\overline{BF}$ and observe that by **SAS**, we have $\triangle BCF \cong \triangle FGB$ (since $\overline{BF}$ is a transversal to paralells $\overleftrightarrow{BG}$ and $\overleftrightarrow{FC}$, so $\measuredangle GBF = \measuredangle CFB$). Thus $|GF| = |BC|$, and the quadrilateral $BGFC$ is a parallelogram, so $\overline{DF} \parallel \overline{BC}$.

In addition, since $\overline{DF}$ is one half of $\overline{GF}$, we also have $|DF| = |BE| = |EC|$.

By an analogous argument, we have $|DE| = |FC| = |AF|$ and $|EF| = |BD| = |AD|$.

Applying **SSS** several times, we obtain $\triangle ADF \cong \triangle DBE \cong \triangle FEC \cong EFD$.

Here is alternative proof, in case you didn't like that one.



Draw perpendiculars to $\overleftrightarrow{BC}$ through points D and F, and construct a parallel to $\overleftrightarrow{BC}$ through A. Call the intersection points of the perpendiculars with the parallel L and R, and let the intersection points with $\overleftrightarrow{BC}$ be M and S.

Then we have $\triangle BDM \cong \triangle LDA$ by **hypoteneuse-angle** ($|BD| = |DA|$ by hypothesis, and we have vertical angles at D); similarly $\triangle ARF \cong \triangle CSF$.

Consequently, $|LD| = |DM| = |RF| = |FS|$, and so quadrilaterals $LRFD$ and $MDFS$ are congurent rectangles. In particular, $\overline{DF} \parallel \overline{BC}$.

A similar argument gives $\overline{AC} \parallel \overline{DE}$ and $\overline{AB} \parallel FE$. From this, we conclude that $|DF| = |EC| = |BE|$, $|EF| = |BD| = |AD|$, and $|DE| = |AF| = |FC|$ (since opposite sides of parallelograms are of equal length).

Since all the corresponding sides agree, we have $\triangle ADF \cong \triangle DBE \cong \triangle FEC \cong EFD$ by **SSS**.

Other variations are certainly possible.