WE LEFT OFF DEFINING

$\int_a^b f(x)\,dx$ AS THE LIMIT OF UPPPER & LOWER SUMS AS THE MESH OF THE PARTITION $\to 0$.

WE SAW f CONTINUOUS $\Rightarrow$ f INTEGRABLE

MORE COMMONLY, WE TAKE A LESS GENERAL DEF.

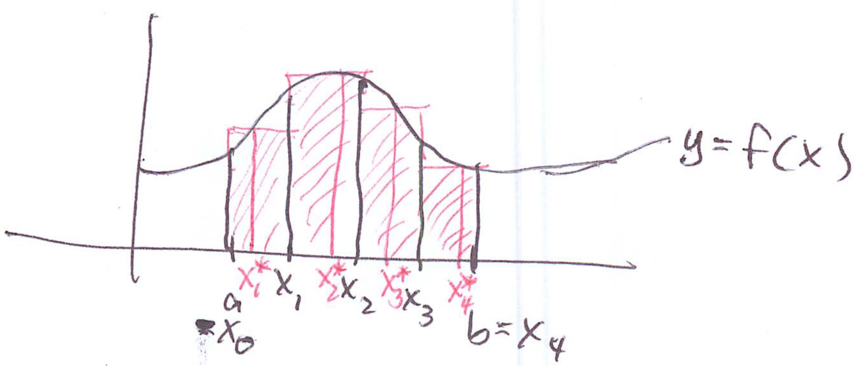
$\cancel{\int_a^b f(x)\,dx =}$

LET $f(x)$ BE CONTINUOUS ON $[a,b]$.

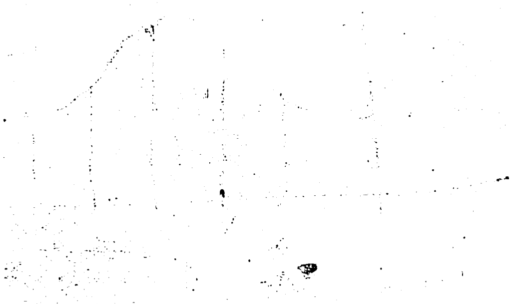
THEN
$$\int_a^b f(x)\,dx = \cancel{\lim_{N\to\infty} \sum_{i=1}^{N} f(x_i^*)(x_i - x_{i-1})}$$
$$= \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i^*)(x_i - x_{i-1})$$

WHERE $a = x_0 < x_1^* < x_1 < x_2^* < x_2 < \cdots < x_N^* < x_N$

AND $(x_i - x_{i-1}) \to 0$ (FOR ALL i) AS $N \to \infty$.



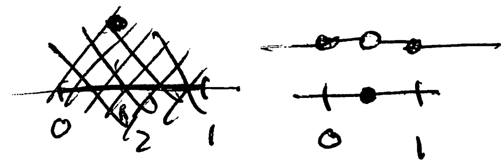
THERE ARE "SPECIAL" VERSIONS WHERE THE x_i^* ARE CHOSEN ON THE RIGHT, LEFT, MIDPOINT, MAX, MIN, ETC.



RETURNING TO THE GENERAL DEF:

SOME DISCONTINUOUS FUNCTIONS ARE INTEGRABLE.

EG $f(x) = \begin{cases} 1 & \text{FOR } x \neq 1/2 \\ 0 & \text{OTHERWISE} \end{cases}$



CERTAINLY HAS $\int_0^1 f(x) = 1$:

OBVIOUSLY, $U(f, \mathcal{P}) = 1$ FOR ANY $\mathcal{P}$.

WHILE $L(f, \mathcal{P}) < 1$ FOR EVERY $\mathcal{P}$, AS THE MESH$(\mathcal{P}) \to 0$,

$L(f, \mathcal{P}) \geq 1 - \varepsilon$ ~~FOR~~ .

THIS CLEARLY GENERALIZES TO ANY FINITE # OF DISCONTINUITIES.

IF THERE ARE INFINITELY MANY, THE SITUATION IS LESS CLEAR. FOR EXAMPLE

$$g(x) = \begin{cases} 1 & \text{FOR } x \in \mathbb{Q} \\ 0 & \text{FOR } x \notin \mathbb{Q} \end{cases}$$

IS NOT INTEGRABLE, SINCE $L(g, \mathcal{P}) = 0$ ON ANY PARTION $\mathcal{P}$ OF $[0,1]$
BUT $U(g, \mathcal{P}) = 1$ ON PARTIONS OF $[0,1]$.

BUT THOMAE'S FUNCTION

$$t(x) = \begin{cases} 1 & \text{IF } x = 0 \\ 1/q & \text{IF } x = p/q \in \mathbb{Q} \setminus \{0\} \text{ IN LEAST TERMS} \\ 0 & \text{IF } x \notin \mathbb{Q} \end{cases}$$

IS INTEGRABLE

THM ~~PROPERTIES~~. • SPOZE $f:[a,b]\to\mathbb{R}$ IS BOUNDED, $c\in[a,b)$.

THEN f IS INTEGRABLE ON $[a,b] \iff f$ INTEGRABLE ON $[a,c]$ AND $[c,b]$, WITH

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

PROOF IS EASY (JUST WRITE PARTIONS OF $[a,b]$ AS THE PARTS IN $[a,c]$ AND $[c,b]$, USE $\varepsilon/2$ INSTEAD OF ε AND ADD ...)

THM SPOZE f, g INTEGRABLE ON $[a,b]$. THEN

- $f\pm g$ INTEGRABLE ON $[a,b]$ WITH $\int_a^b f\pm g = \int_a^b f \pm \int_a^b g$
- FOR $k\in\mathbb{R}$, $\int_a^b kf = k\int_a^b f$
- IF $m\le f(x)\le M$ ON $[a,b]$ THEN $m(b-a)\le \int_a^b f \le M(b-a)$
- IF $f(x)\le g(x)$ ON $[a,b]$ THEN $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$
- $|f|$ IS INTEGRABLE AND $\left|\int_a^b f\right| \le \int_a^b |f|$

DEF FOR CONVENIENCE OF ALGEBRA, ~~WE DEFI~~ IF f IS INTEGRABLE ON $[a,b]$, DEFINE

$$\int_b^a f = -\int_a^b f \quad \text{AND} \quad \int_c^c f = 0 \quad (c\in[a,b])$$

COMMENT: WE DISCUSSED UNIFORM CONVERGENCE OF FUNCTIONS BRIEFLY BUT SPOZE $f_n(x)\to f(x)$ ON $[a,b]$

DOES $\int_a^b f_n \to \int_a^b f$? SOMETIMES.

IF $f_n\to f$ UNIFORMLY (IE δ DOESNT DEPEND ON x, ONLY ε) THEN YES.

CONSIDER THE FOLLOWING:

$$f_n(x) = \begin{cases} n & \text{IF } 0 \le x \le 1/n \\ 0 & \text{OTHERWISE.} \end{cases}$$

FOR EACH n $\left.\begin{array}{l} \int_0^1 f_n(x)\,dx = 1, \\ \text{SO } \lim \int_0^1 f_n(x)\,dx = 1 \end{array}\right)$ BUT $\lim_{n\to\infty} f_n(x) \to 0$

(AND IS UNDEFINED AT $x=0$)

SO WE HAVE $\int_0^1 \lim f_n = 0 \neq \lim \int_0^1 f_n = 1$.

THE FUNDAMENTAL THEOREM OF CALCULUS

THM (FTC):

(i) IF $f:[a,b] \to \mathbb{R}$ IS INTEGRABLE ON $[a,b]$ AND THERE IS $F:[a,b] \to \mathbb{R}$ WITH $F'(x) = f(x)$ ON $[a,b]$ THEN

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

(ii) LET $g:[a,b] \to \mathbb{R}$ BE INTEGRABLE, AND FOR EACH $x \in [a,b]$ LET

$$G(x) = \int_a^x g(t)\,dt$$

THEN G IS CONTINUOUS ON $[a,b]$.

IF g IS CONTINUOUS AT $c \in [a,b]$, THEN G IS DIFFERENTIABLE AT c AND $G'(c) = g(c)$.

Pf of i) Let P be a partition of $[a,b]$. On each interval, apply the MVT to get $x_i^* \in [x_i, x_{i+1}]$ with

$$F(x_{i+1}) - F(x_i) = F'(x_i^*)(x_{i+1} - x_i) = f(x_i^*)(x_{i+1} - x_i)$$

Since each $f(x_i^*)$ is bounded by

$$m_i = \inf_{x_i < x < x_{i+1}} f(x) \leq f(x_i^*) \leq \sup_{x_i < x < x_{i+1}} f(x) = M_i$$

we have

$$L(f,P) \leq \sum_{i=0}^{n} \big(F(x_{i+1}) - F(x_i)\big) \leq U(f,P).$$

But the middles cancel to get $F(x_n) - F(x_0)$
$= F(b) - F(a)$.

Pf of ii) Take $x > y$ on $[a,b]$, and note

$$|G(x) - G(y)| = \left| \int_a^x g - \int_a^y g \right| = \left| \int_y^x g \right| \leq \int_y^x |x| \leq M(x-y)$$

where $|g(x)| < M$ on $[a,b]$.
~~Assuming g is continuous on [a,b]~~, since G is bounded on a compact interval, this shows it is uniformly continuous.

~~Now~~

→

ASSUMING g IS CONTINUOUS AT $c \in [a,b]$, WE WANT $G'(c) = g(c)$.

$$\lim_{x \to c} \frac{G(x)-G(c)}{x-c} = \lim_{x \to c} \frac{1}{x-c}\left(\int_a^x g(t)\,dt - \int_a^c g(t)\,dt\right)$$
$$= \lim_{x \to c} \frac{1}{x-c} \int_c^x g(t)\,dt.$$

TO SEE THIS IS $g(c)$, FIX $\varepsilon > 0$. WE NEED δ SO THAT

$$|x-c| < \delta \Rightarrow \left|\left(\frac{1}{x-c}\int_c^x g(t)\,dt\right) - g(c)\right| < \varepsilon$$

BUT SINCE g CONTINUOUS, $|t-c| < \delta \Rightarrow |g(t)-g(c)| < \varepsilon$.

THIS MEANS WE CAN WRITE $g(c) = \frac{1}{x-c}\int_c^x g(c)\,dt$ ← CONSTANT!

SO (ASSUMING $|x-c| \geq |t-c|$)

$$\left|\frac{1}{x-c}\left(\int_c^x g(t)\,dt\right) - g(c)\right| = \left|\frac{1}{x-c}\left(\int_c^x g(t)-g(c)\,dt\right)\right|$$
$$\leq \frac{1}{x-c}\int_c^x |g(t)-g(c)|\,dt$$
$$< \frac{1}{x-c}\int_c^x \varepsilon\,dt = \varepsilon.$$

☺