LET'S COMPARE

$$\lim_{x \to c} f(x) \quad \text{WITH} \quad f(x) = 3x+1$$

AND $\lim_{x \to c} g(x)$, $g(x) = x^2$.

IN THE FIRST CASE, LET $\delta = \varepsilon/3$.

$$|f(x)-f(c)| = |3x+1-(3c+1)| = 3|x-c|$$

SO $|x-c| < \delta \Rightarrow |f(x)-f(c)| < 3\,\varepsilon/3 = \varepsilon$.

FOR x^2, THE CHOICE DEPENDS ON c:

$$|g(x)-g(c)| = |x^2-c^2| = |x-c|\cdot|x+c|.$$

WE SAW THAT AS LONG AS $\delta < 1$, $x \in (c-1, c+1)$

$$|x+c| \le |x|+|c| \le (|c|+1)+|c| = 2|c|+1$$

SO TAKING $\varepsilon > 0$, $\delta = \min\left\{1, \frac{\varepsilon}{2|c|+1}\right\}$ DOES THE JOB:

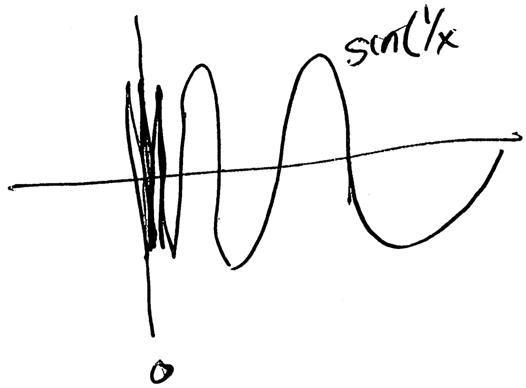
$$|f(x)-f(c)| = |x-c|\cdot|x+c| < \left(\frac{\varepsilon}{2|c|+1}\right)(2|c|+1) = \varepsilon.$$

THAT IS, LARGE VALUES OF c REQUIRE INCREASINGLY SMALL VALUES OF δ.

DEF $f: A \to \mathbb{R}$ IS UNIFORMLY CONTINUOUS ON A IF FOR EVERY $\varepsilon > 0$, THERE IS A $\delta > 0$ SO THAT FOR ALL $x, y \in A$,

$$|x-y| < \delta \Rightarrow |f(x)-f(y)| < \varepsilon$$

EX: $h(x) = \sin(1/x)$ IS CONTINUOUS AT EVERY $x \in (0,1)$, BUT IT IS NOT UNIFORMLY CONTINUOUS



NEAR 0, A SMALL INTERVAL $(c-\varepsilon, c+\varepsilon)$ IS SENT TO A LARGE ONE, ~~ALMOST~~

THM: IF f IS CONTINUOUS ON A COMPACT SET K, THEN f IS UNIFORMLY CONTINUOUS ON K

Pf/ SPOZE $f : K \to \mathbb{R}$ IS CONTINUOUS AT EVERY $x \in K$.

SPOZE FOR CONTRADICTION f IS NOT UNIF. CONT ON K. THEN THERE ARE SEQUENCES $\{x_n\}$ AND $\{y_n\}$ IN K SO THAT

$$\lim |x_n - y_n| = 0 \quad \text{BUT} \quad |f(x_n) - f(y_n)| > \varepsilon_0$$

FOR SOME ε_0.

BUT SINCE K IS COMPACT THERE IS A CONVERGENT SUBSEQUENCE x_{n_k} WITH $x_{n_k} \to x_* \in K$

LOOK AT CORRESPONDING y_{n_k}.

$$\lim y_{n_k} = \lim((y_{n_k} - x_{n_k}) + x_{n_k}) = 0 + x_*$$

SO x_{n_k} AND y_{n_k} HAS SAME LIMIT. $x_* \in K$

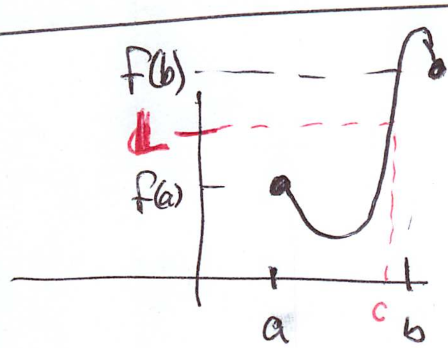
SINCE f IS CONTINUOUS,

$$\lim f(x_{n_k}) = f(x_*) = \lim f(y_{n_k})$$

$$\Rightarrow \lim(f(x_{n_k}) - f(y_{n_k})) = 0 \quad \Rightarrow\Leftarrow$$

THE INTERMEDIATE VALUE THEOREM (IVT).

LET $f : [a,b] \to \mathbb{R}$ BE CONTINUOUS. IF $L \in \mathbb{R}$ SATISFIES $f(a) < L < f(b)$ OR $f(b) < L < f(a)$ THEN THERE EXISTS $c \in (a,b)$ FOR WHICH $f(c) = L$



THIS THEOREM IS A "TOTALLY OBVIOUS" FACT". IT WAS FREELY USED WITHOUT PROOF IN THE 18^{TH} CENTURY, BUT WASN'T PROVEN UNTIL 1817 (BY BOLZANO), MOSTLY BECAUSE THE IDEAS LIKE CONTINUITY HADN'T BEEN FORMALIZED, NOR THE NATURE OF THE REAL NUMBERS.

HERE IS ONE APPROACH:

THM: LET $f : A \to \mathbb{R}$ BE CONTINUOUS, AND LET $E \subseteq G$ BE CONNECTED. THEN $f(E)$ IS CONNECTED.

THIS PROVES THE IVT.

OH... WHAT DOES IT MEAN FOR A SUBSET OF $\mathbb{R}$ TO BE CONNECTED? IT IS "OBVIOUS": THERE ARE NO "GAPS". BUT WHAT DOES THAT MEAN?

IT CAN GET TRICKY, ALTHOUGH IN $\mathbb{R}$ THINGS ARE SIMPLER:

DEF LET A, B BE SUBSETS OF $\mathbb{R}$.

- A & B ARE SEPARATED IF $\bar{A} \cap B = \emptyset = A \cap \bar{B}$
- $E \subseteq \mathbb{R}$ IS DISCONNECTED IF $E = A \cup B$ WHERE A & B ARE NONEMPTY, SEPARATED SETS.
- A ~~SUBSET~~ SET IS CONNECTED IF IT IS NOT DISCONNECTED.

THM: A SET $E \subseteq \mathbb{R}$ IS CONNECTED $\iff$ WHENEVER $a \in E$, $b \in E$ WITH $a < b$ AND $a < c < b$, THEN $c \in E$ (IN OTHER WORDS, E IS AN INTERVAL)

Pf [SKIP IN CLASS]

$\Rightarrow$/ SPOZE E IS CONNECTED WITH $a, b \in E$, $a < c < b$.

LET $A = (-\infty, c) \cap E$ $\quad B = (c, \infty) \cap E$.

$A \neq \emptyset$, $B \neq \emptyset$. NO LIMIT POINT OF A CAN BE IN B, NOR CAN ANY LIMIT POINT OF B BE IN C.

THUS $\bar{A} \cap B = \emptyset = A \cap \bar{B}$. IF $E = A \cup B$, IT IS DISCONNECTED. THUS $c \in E$.

$\Leftarrow$ SPOZE E IS AN INTERVAL, AND $E = A \cup B$. LET $a_0 \in A$, $b_0 \in B$ AND WLOG $a_0 < b_0$. LET $I_0 = [a_0, b_0]$. CONSIDER $\left[a_0, \frac{a_0+b_0}{2}\right]$ AND $\left[\frac{a_0+b_0}{2}, b_0\right]$. LET I_1 BE WHICHEVER OF THESE $= [a_1, b_1]$

~~[scribbled]~~ HAS $a_1 \in A$, $b_1 \in B$, IE $I_1 = \begin{cases} [a_0, \frac{a_0+b_0}{2}] & \text{IF } \frac{a_0+b_0}{2} \in B \\ [\frac{a_0+b_0}{2}, b_0] & \text{IF NOT.} \end{cases}$

CONTINUE IN THIS WAY TO GET

$I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$. SINCE $|a_n - b_n| \to 0$,

$\bigcap_{n=0}^{\infty} I_n = \{x\}$. BUT ALSO $\lim\{a_n\} = x = \lim\{b_n\}$

SINCE $x \in E = A \cup B$, EITHER $x \in A$ OR $x \in B$,

SO E IS CONNECTED.

BACK TO PROVING THAT "f CONTINUOUS, E CONNECTED $\Rightarrow$ $f(E)$ IS CONNECTED".

~~SPOZE NOT, IE~~ SPOZE, $f(E) = A \cup B$ WITH $A \cap B = \emptyset$, $A \neq \emptyset$, $B \neq \emptyset$.

LET $C = \{x \in E \mid f(x) \in A\} = f^{-1}(A) \cap E$ ← PREIMAGE OF A

$D = \{x \in E \mid f(x) \in B\} = f^{-1}(B) \cap E$ ← PREIMAGE OF B.

NOW $C \neq \emptyset$ ~~[scribbled]~~ $D \neq \emptyset$ AND $C \cap D = \emptyset$.

IE DISJOINT, NONEMPTY.

BY ASSUMPTION, E WAS CONNECTED. ~~THUS,~~

~~EXISTS SEQUENCES THUS, WE CAN FIND A SEQ~~

THIS MEANS EITHER $\overline{C} \cap D$ IS NONEMPTY OR $\overline{D} \cap C \neq \emptyset$.

WLOG, ASSUME THE FIRST. THEN THERE IS A SEQUENCE $\{x_n\}$ WITH $x_n \in C$ AND ~~$\lim x_n \in D$~~. $\lim x_n = x \in D$.

SINCE f IS ~~CONNECTED~~, CONTINUOUS,

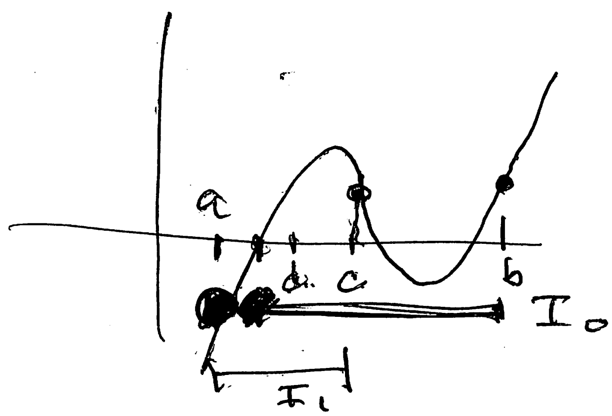
$$\lim f(x_n) = f(x)$$

BUT $f(x_n) \in A$ AND $f(x) \in B$, THAT IS $\overline{A} \cap B \neq \emptyset$

IE $f(E)$ CONNECTE

THE INTERMEDIATE VALUE THEOREM TELLS US THAT IF f IS A CONTINUOUS FUNCTION WITH $f(a) < 0$ AND $f(b) > 0$, $f(x) = 0$ HAS AT LEAST ONE SOLUTION.

WE CAN DO THIS CONSTRUCTIVELY; USING THE BISECTION METHOD (AND IVT)



LET $I_0 = [a, b]$, WITH $f(a)$ $f(b)$ HAVING DIFF SIGNS.

LET $c = \frac{a+b}{2}$.

IF $f(c)$ HAS THE SAME SIGN AS $f(a)$,

LET $I_1 = [c, b]$ ELSE $I_1 = [a, c]$.

CONTINUE IN THIS WAY TO GET NESTED INTERVALS...