- RETURN MIDTERM, DISCUSS A BIT

(MEDIAN = 28/60. 4 "GOOD" SCORES (≥ 45)
1 "OK" SCORE (≥ 35)
6 "LESS GOOD" (< 30))

ME = ☹

OPEN / CLOSED / COMPACT

WE'VE DEALT WITH OPEN INTERVALS LIKE $(0,1) = \{x \mid 0 < x < 1\}$ A FAIR AMOUNT. THE MAIN THING ABOUT OPEN SETS IS THAT POINTS CAN JIGGLE AROUND AND STAY INSIDE. IN OTHER WORDS, A SET IS OPEN IF EVERY POINT HAS A NEIGHBORHOOD WHICH IS FULLY IN THE SET.

RECALL: THE <u>ε-NEIGHBORHOOD OF $a \in \mathbb{R}$</u> IS

$$V_\varepsilon(a) = \{ x \in \mathbb{R} \mid |x-a| < \varepsilon \} = (a-\varepsilon, a+\varepsilon)$$

<u>DEF</u> A SET $O \subseteq \mathbb{R}$ IS <u>OPEN</u> IF FOR EVERY POINT $x \in O$, THERE IS AN ε SO THAT $V_\varepsilon(x) \subset O$

EXAMPLES:

- $\mathbb{R}$ IS OPEN.
- $\emptyset$ IS OPEN (BECAUSE THERE IS NO $x \in \emptyset$)
  SO FOR EVERY x ... HOLDS
- EVERY OPEN INTERVAL (c,d) IS OPEN.
- THE UNION OF TWO OPEN INTERVALS IS OPEN.

THM: THE UNION OF ~~OF~~ AN ARBITRARY COLLECTION OF OPEN SETS IS OPEN.

NOTE THAT THE ~~[illegible]~~ "NUMBER" OF SETS COULD BE UNCOUNTABLE.

Pf/ LET $\mathcal{O}_\lambda$ BE OPEN FOR EVERY $\lambda \in \Lambda$,

AND LET $\mathcal{O} = \bigcup_{\lambda \in \Lambda} \mathcal{O}_\lambda$.

NOW SPOZE $a \in \mathcal{O}$. SINCE THERE IS

SOME λ_0 SO THAT $a \in \mathcal{O}_{\lambda_0}$, AND $\mathcal{O}_{\lambda_0}$ IS OPEN,

THERE IS A NEIGHBORHOOD $V_\varepsilon(a) \subset \mathcal{O}_{\lambda_0}$.

BUT THEN $V_\varepsilon(a) \subset \mathcal{O}$, SO $\mathcal{O}$ IS OPEN: EVERY a HAS A NBHD.

$\underline{\text{THM}}$: THE INTERSECTION OF $\underline{\text{FINITELY MANY}}$ ~~SE~~ OPEN SETS IS OPEN

(NOTE: FALSE FOR ∞-MANY. CONSIDER

$\bigcap_{n=1}^{\infty} (0, 1+1/n)$. THIS IN NOT OPEN, SINCE THERE IS NO NEIGHBORHOOD OF 1 IN THE INTERSECTION.)

Pf/ SPOZE $\{\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_N\}$ IS A $\underline{\text{FINITE}}$ COLLECTION OF OPEN SETS.

LET $\mathcal{O} = \bigcap_{k=1}^{N} \mathcal{O}_k$, AND $a \in \mathcal{O}$.

SINCE $a \in \mathcal{O}$, WE KNOW $a \in \mathcal{O}_k$ FOR EACH k;

THAT IS THERE ARE NEIGHBORHOODS $V_{\varepsilon_k}(a) \subseteq \mathcal{O}_k$ FOR EACH k.

NOW LET $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k\}$.

SINCE $V_\varepsilon(a) \subseteq V_{\varepsilon_k}(a)$ FOR EACH k, WE HAVE $V_\varepsilon(a) \subset \mathcal{O}$ FOR ANY $a \in \mathcal{O}$. THUS $\mathcal{O}$ IS OPEN.

DEF: A POINT x IS A $\left\{\begin{array}{l}\text{CLUSTER POINT} \\ \text{ACCUMULATION POINT} \\ \underline{\text{LIMIT POINT}}\end{array}\right\}$ OF A SET A IF EVERY ε-NEIGHBORHOOD OF ~~A~~ x INTERSECTS A AT SOME POINT $y \neq x$.

THE WORD "LIMIT POINT" IMPLIES THERE IS SOMETHING TO DO WITH LIMITS HERE!

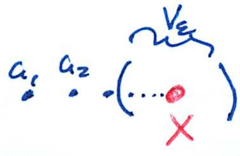
THM: A POINT x IS A LIMIT POINT OF THE SET A
$\Longleftrightarrow$ THERE IS A SEQUENCE $\{a_n\}$ WITH $a_n \in A$, $a_n \neq x$
SO THAT $x = \lim a_n$.

Pf/ $\Rightarrow$ SPOZE x IS A LIMIT POINT OF A. WE NEED TO FIND A SEQUENCE $\{a_n\}$ IN A SO THAT $a_n \to x$ WITH $a_n \neq x$ FOR ALL n.

CONSIDER $V_{1/n}(x)$. SINCE x IS A LIMIT POINT, THE SET $\left(V_{1/n}(x) \cap A\right) - \{x\}$ IS NOT EMPTY, SO LET a_n BE A POINT IN THAT SET.

BUT THEN FOR ANY $\varepsilon > 0$, $\exists\, K \in \mathbb{N}$ SO THAT $1/n < \varepsilon$ FOR $n > K$
SO $|x - a_n| < 1/n < \varepsilon$ AND $a_n \to x$.

$\Leftarrow$/ SPOZE $\lim a_n = x$, WITH $a_n \in A$, $a_n \neq x$.
BUT THEN FOR ANY $\varepsilon > 0$, WE HAVE POINTS $a_n \in V_\varepsilon(x)$ & WITH $a_n \in A$, $a_n \neq x$.



SO x IS A LIMIT POINT OF A.

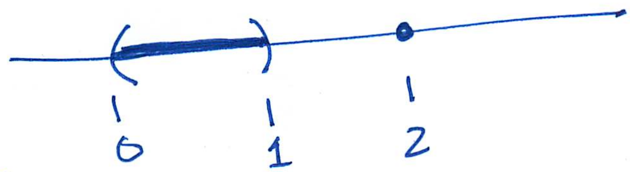
THE REASON FOR THE RESTRICTION $a_n \neq x$ IS TO RULE OUT A CONSTANT SEQUENCE LIKE $x, x, x, \ldots$

SUCH POINTS ARE ISOLATED

DEF A POINT $a \in A$ IS AN ISOLATED POINT OF A IF IT IS NOT A LIMIT POINT OF A.

NOTE THAT IF a IS AN ISOLATED POINT OF A, WE HAVE $a \in A$. BUT IF x IS A LIMIT POINT OF A, PERHAPS $x \in A$ AND PERHAPS NOT.

EXAMPLE: $A = (0,1) \cup \{2\}$. THEN 2 IS ISOLATED, 1 IS A LIMIT POINT (AS IS 0 AND EVERY OTHER POINT OF A)



ANOTHER: EVERY $x \in \{\frac{1}{n} \mid n \in \mathbb{N}\}$ IS ISOLATED

DEF: A SET $E \subseteq \mathbb{R}$ IS CLOSED IF IT CONTAINS ALL OF ITS LIMIT POINTS.

(IN MATH WE USE THE TERM "CLOSED" A LOT TO MEAN THAT SOME OPERATION ON ELEMENTS OF THE SET HAS A RESULT THAT IS STILL IN THE SET.

EG: A VECTOR SPACE IS CLOSED UNDER ADDITION AND SCALAR MULTIPLICATION. $\mathbb{R}$ IS CLOSED UNDER ADDITION AND MULT. BUT NOT DIVISION.

HERE, CLOSED MEANS CLOSED UNDER TAKING LIMITS OF ELTS. IN THE SET)

DEF: A SET $K \subseteq \mathbb{R}$ IS COMPACT IF EVERY SEQUENCE OF ELEMENTS IN K HAS A SUBSEQUENCE THAT CONVERGES TO A LIMIT IN K

EXAMPLES:
- ANY CLOSED INTERVAL $[a,b]$ IS COMPACT.
- THE CANTOR SET $\mathcal{C}$ IS COMPACT.
- $\mathbb{R}$ IS CLOSED, BUT NOT COMPACT.

  ($\mathbb{R}$ IS NOT COMPACT SINCE A SEQUENCE LIKE $\{n\}_{n=1}^{\infty}$ HAS NO CONVERGENT SUBSEQUENCE.
  HOWEVER, IT IS CLOSED: ANY CONVERGENT SEQUENCE OF REALS CONVERGES TO A LIMIT IN $\mathbb{R}$).
- $\{\frac{1}{n} \mid x \in \mathbb{N}\} \cup \{0\}$ IS CLOSED.

THM: A SET $E \subseteq \mathbb{R}$ IS CLOSED $\iff$ EVERY CAUCHY SEQUENCE OF POINTS $e_n \in E$ CONVERGES TO A LIMIT $e \in E$.

PF/ HW.