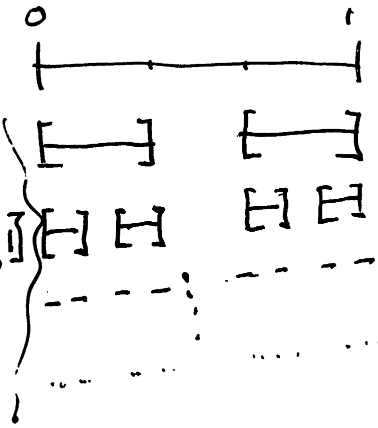
MAT 513 3/2/22

1

THE CANTOR SET

LETS START WITH $\mathcal{C}_0 = [0,1]$



LET $\mathcal{C}_1 = \mathcal{C}_0 \setminus (\frac{1}{3}, \frac{2}{3}) = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$

$\mathcal{C}_2 = \left([0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}]\right) \cup \left([\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]\right)$

AT EACH STAGE REMOVE THE OPEN INTERVAL WHICH IS THE MIDDLE THIRD OF EACH INTERVAL IN THE PREVIOUS STAGE.

SO $\mathcal{C}_n$ IS THE UNION OF 2^n CLOSED INTERVALS EACH OF LENGTH $1/3^n$.

THE MIDDLE-THIRDS CANTOR SET

$$\mathcal{C} = \bigcap_{n=0}^{\infty} \mathcal{C}_n$$

IT MAY SEEM LIKE VERY LITTLE IS LEFT AFTER THIS PROCESS:

$$\mathcal{C} = [0,1] \setminus \left\{ (\tfrac{1}{3}, \tfrac{2}{3}) \cup (\tfrac{1}{9}, \tfrac{2}{9}) \cup (\tfrac{7}{9}, \tfrac{8}{9}) \cup \cdots \right\}$$

BUT IT CERTAINLY ISN'T EMPTY, SINCE

$$0 \in \mathcal{C},\ 1 \in \mathcal{C},\ \{\tfrac{1}{3}, \tfrac{1}{9}, \tfrac{1}{27}, \cdots\} \subset \mathcal{C}\ \{\tfrac{2}{3}, \tfrac{2}{9}, \tfrac{2}{27}, \cdots\} \subset \mathcal{C}$$

CERTAINLY ... IS $\mathcal{C}$ COUNTABLE?

$\mathcal{C}$ IS AN INFINITE SET

NOTE THAT WE'VE REMOVED "ALL" THE LENGTH OF $\mathcal{C}$:

WE START WITH A SEGMENT OF LENGTH 1,

REMOVED ONE OF LENGTH $1/3$
TWO OF LENGTH $1/9$
FOUR OF LENGTH $1/27$
$\vdots$

THAT IS WE'VE TAKEN OUT A TOTAL LENGTH OF

$$\sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}} = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{3} \cdot \frac{1}{1 - 2/3} = 1.$$

IE, "BASICALLY" EVERYTHING.

BUT IN FACT $\mathcal{C}$ HAS THE SAME CARDINALITY AS $\mathbb{R}$..!

HOW TO SEE THIS?

LET'S TRY TO LIST ALL THE POINTS OF $\mathcal{C}$ IN A CONCISE WAY.

- FIRST, LETS WRITE EVERY NUMBER $0 < x < 1$ IN BASE 3, ~~THAT IS~~, DECIMALS (TRICIMALS?)

FOR EXAMPLE, $\frac{1}{3} = 0.1000\overline{0}\ldots$

$$\frac{47}{81} = \frac{1}{3} + \frac{2}{9} + \frac{0}{27} + \frac{2}{81} = 0.1202$$

$$0.111\overline{1}\ldots = \sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{3}\left(\frac{1}{1-1/3}\right) = \frac{1}{3}\left(\frac{3}{2}\right) = \frac{1}{2}.$$

IN GENERAL, WE WRITE EVERY NUMBER IN $[0,1]$

AS $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{3^n}$ WHERE $a_n \in \{0,1,2\}$

NOTE THAT IN BASE 3, WE HAVE THE SAME ISSUE WITH NUMBERS ENDING IN ALL 2s:

$$0.22\overline{2}\ldots = 1.\overline{0}$$

SINCE $0,2\overline{2} = \frac{1}{3}\left(\sum_{n=0}^{\infty} \frac{2^n}{3^n}\right) = \frac{1}{3} \frac{1}{1-2/3} = 1.$

SO WHATS THE POINT?

~~WHAT DOES "DELETING THE~~

C_0 = ALL POINTS IN $[0,1]$ = #'s OF THE FORM $0.$ ANY

$C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ = ALL NUMBERS ~~WITH~~ OF THE FOR 0.0ANY... AND 0.2ANY... (IE NO 1 IN THE 1^{ST} PLACE)

$$C_2 = \{0.a_1a_2a_3\ldots \mid a_1, a_2 \in \{0,2\}\}.$$

AND IN GENERAL

$$C = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \text{ WITH } a_n \in \{0,2\}. \quad (\text{IE NO 1s.})$$

RECALL THAT WE CAN WRITE

$\frac{1}{3}$ AS $0.02\overline{2}$.

SO AN ALTERNATIVE DEFINITION OF C IS

> THE SET OF ALL NUMBERS ~~WHO~~ WHICH CAN BE REPRESENTED AS A "TERNARY DECIMAL" USING NO 1s.

NOW, TO SEE THAT C AND $\mathbb{R}$ HAVE THE SAME CARDINALITY, WE NEED A BIJECTION $f: C \to \mathbb{R}$. IN FACT $f: C \to (0,1)$ IS SUFFICIENT, SINCE WE KNOW $(0,1) \leftrightarrow \mathbb{R}$ VIA $x \mapsto \frac{x - 1/2}{x(x-1)}$

NOW CONSIDER THE FOLLOWING $f:(0,1)\to(0,1)$.

IF $x = 0.a_1a_2a_3a_4\cdots = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$

LET $b_n = \begin{cases} 0 & \text{IF } a_n = 0 \\ 1 & \text{IF } a_n = 2 \end{cases}$

LET $f(x) = \sum_{n=1}^{\infty} \frac{b_n}{2^n} = 0.b_1b_2b_3\cdots$ [BASE 2].

SINCE EVERY REAL NUMBER IN $(0,1)$ HAS A UNIQUE BINARY DECIMAL REPRESENTATION (EXCEPT FOR THE ISSUE OF $1.00\overline{0} = 0.11\overline{1}$ ~~BUT WE~~) AND ALL POSSIBLE STRINGS OF 0 & 1 OCCUR.

THIS MAP IS A SURJECTION.

IT IS NOT AN INJECTION,

SINCE $f(\frac{1}{3}) = f(.022\overline{2}) = .011\overline{1}_2 = \frac{1}{2}$

$f(\frac{2}{3}) = f(.2\overline{0}) = .1\overline{0}_2 = \frac{1}{2}$.

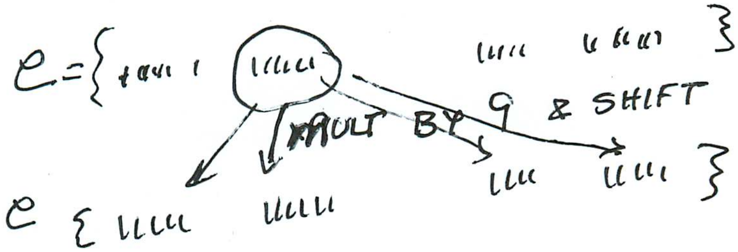
BUT THAT'S OK: THERE ARE ONLY COUNTABLY MANY SUCH POINTS.

SO $\text{card}(\mathcal{C}) = \text{card}((0,1)) = \mathfrak{c}$.

THIS SET C IS SELF-SIMILAR: THERE IS AN AFFINE MAP WHICH MAPS SMALL PARTS OF THE SET ONTO THE WHOLE SET.

FOR EXAMPLE: $x \longmapsto 3x$ IS A BIJECTION BETWEEN $C \cap [0, \frac{1}{3}]$ AND C.

BLOWING UP ANY SMALL INTERVAL GIVES BACK THE WHOLE SET.



<u>DIMENSION</u>

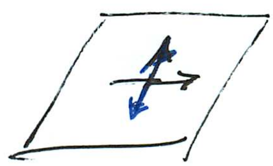
A POINT (OR A COUNTABLE COLLECTION OF POINTS

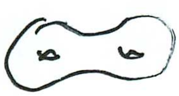
HAS DIMENSION 0

A LINE SEGMENT (OR A CURVE) HAS DIMENSION 1

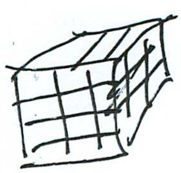
A PLANE OR SURFACE HAS DIMENSION 2

SO DOES A SURFACE (SPHERE, ...)





SPACE HAS DIMENSION 3



ETC.

BASIC IDEA IS NUMBER OF COORDINATES YOU CAN VARY LOCALLY AND STAY IN THE SET.

THE CANTOR SET (EXTENDING THE DEFINITION OF DIMENSION)

HAS DIMENSION $\dfrac{\log 2}{\log 3} = 0.6309\ldots$

IT IS A **FRACTAL**

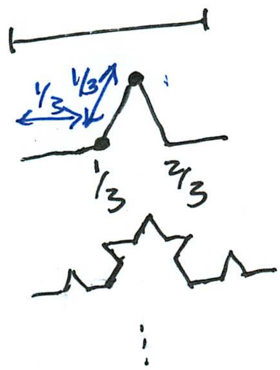
WHAT DOES A FRACTIONAL DIMENSION EVEN MEAN?

ITS A LITTLE EASIER TO LOOK AT A DIMENSION BETWEEN 1 AND 2, CONSTRUCTED IN A SIMILAR WAY.

THE KOCH CURVE (HELGE VON KOCH, 1904)

THIS IS AN "INFINITELY LONG" "CURVE" THAT FITS IN A SMALL ~~SPACE~~. AREA.

IT IS "TOO BIG" TO BE PARAMETERIZED BY ONE NUMBER, BUT DOES NOT FILL SPACE, SO THE DIMENSION SHOULD BE LESS THAN 2.



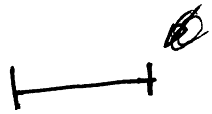
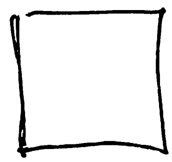
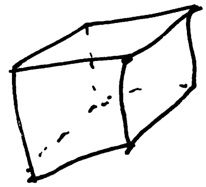
DIMENSION $= \dfrac{\log 4}{\log 3} = 1.26186\ldots$

(REPLACE EVERY ___ WITH _/_)

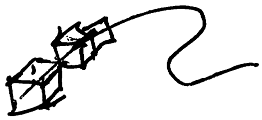
LENGTH $1 \to 4/3$

~~BUT~~ THE BEST WAY TO DESCRIBE THIS IS THE HAUSDORFF DIMENSION WHICH IS BEYOND THE SCOPE OF THIS COURSE. BUT FOR SELF-SIMILAR SETS, WE CAN DEFINE "COVERING DIMENSION" OR SCALING DIMENSION.

LETS START WITH SOME KNOWN EXAMPLES.

DIM 1	DIM 2	DIM 3
		

~~SUPPOSE~~ YOUR SET LIVES IN SOME AMBIENT EUCLIDEAN SPACE, AND YOU COUNT HOW MANY "BOXES" OF DIAMETER ε YOU NEED TO COVER IT, AND HOW THIS DEPENDS ON ε

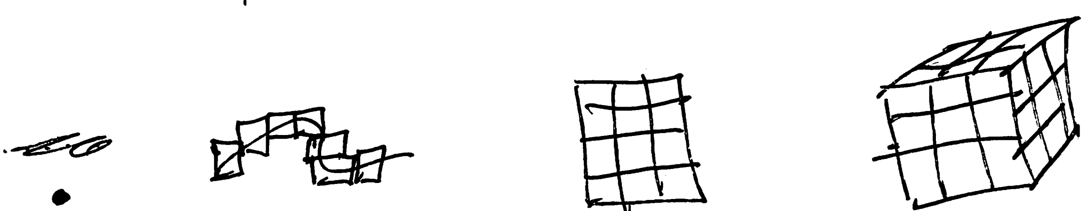


LET $N(\varepsilon)$ BE THE NUMBER NEEDED FOR ~~A K-DIM~~ ~~OBJECT~~ TO COVER IT.

THEN $$\mathrm{DIM}_{\mathrm{BOX}}(S) = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{-\log \varepsilon}$$

NOTE THAT FOR A CURVE, THE NUMBER NEEDED GROWS LIKE $\frac{1}{\varepsilon}$, FOR A SURFACE $\frac{1}{\varepsilon^2}$, FOR A VOLUME LIKE $\frac{1}{\varepsilon^3}$.

FOR THE CANTOR SET, IT IS $\frac{\log 2}{\log 3} = 0.63\ldots$



CONSTANT. n MANY n^2 MANY n^3 MANY.