# CARDINALITY

FOR FINITE SETS, WE CAN COUNT THE NUMBER OF ELEMENTS. WHAT IS COUNTING?
JUST ASSIGNING A ~~NAT~~ NATURAL NUMBER TO EACH ELEMENT OF THE SET, IN ORDER.

$$\begin{array}{ccc} \{7, & \text{CAT}, & \emptyset\} \\ \updownarrow & \updownarrow & \updownarrow \\ 1 & 2 & 3. \end{array}$$

OF COURSE, WE DONT HAVE TO ACTUALLY ~~COUNT~~ DO THE ASSIGNMENT, AS LONG AS THERE IS A BIJECTION: THE NUMBER OF NOSES IN THE ROOM IS THE SAME AS THE NUMBER OF PEOPLE.

SO THIS IS REALLY $f: \{\text{NOSES}\} \longleftrightarrow \{\text{PEOPLE}\}$

IE $|\{\text{NOSES}\}| = |\{\text{PEOPLE}\}|$ (WHERE $|\cdot|$ = "SIZE OF" = "CARDINALITY")

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**DEF:** TWO SETS $A$ & $B$ HAVE THE SAME CARDINALITY IF THERE IS A BIJECTION $|A| \longleftrightarrow |B|$, IE $f: A \to B$.
WRITE $|A| = |B|$ OR $A \sim B$ OR $\text{card}(A) = \text{card}(B)$

EXAMPLE: LET $E = \{\text{EVEN NATURALS}\} = \{2, 4, 6, 8, 10, \cdots\}$.
THEN $E \sim \mathbb{N}$ SINCE $f: \mathbb{N} \to \mathbb{Z}$ WITH $f(n) = 2n$ IS A BIJECTION.

- SIMILARLY, $\mathbb{Z} \sim \mathbb{N}$ SINCE

$$f(n) = \begin{cases} 2n & \text{IF } n > 0 \\ -2n+1 & \text{IF } n \leq 0 \end{cases}$$

$$\begin{array}{cccccccccc} \mathbb{Z} = \{ & \dots, & -3, & -2, & -1, & 0, & 1, & 2, & 3, & \dots \} \\ & & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ \mathbb{N} = \{ & \dots, & 7, & 5, & 3, & 1, & 2, & 4, & 6, & \dots \} \end{array}$$

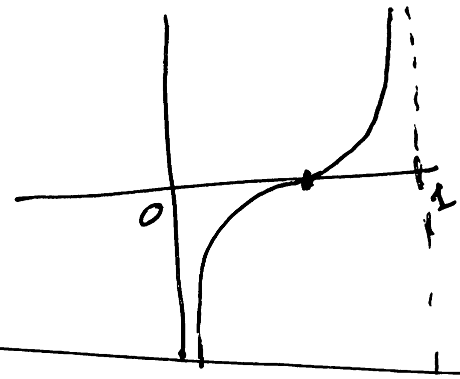
(IF A SET A IS THE SAME CARDINALITY AS A PROPER SUBSET, BOTH MUST BE INFINITE)

- $\mathbb{R} \sim (0,1)$.

  LET ~~f~~ $f: (0,1) \to \mathbb{R}$ WITH $f(x) = \dfrac{x - 1/2}{x(x-1)}$

  f IS 1-1, ONTO.

  [IN FACT $\mathrm{card}((a,b)) = \mathrm{card}(\mathbb{R})$ FOR $a < b$.




DEF:

- IF $|A| = n$ FOR $n \in \mathbb{N} \cup \{0\}$, A IS FINITE
- IF $|A| = |\mathbb{N}| = \aleph_0$ (ALEPH-NULL OR ALEPH-NOUGHT), A IS COUNTABLY INFINITE (OR "COUNTABLE")
- OTHERWISE A IS UNCOUNTABLE

- SOME PEOPLE USE "COUNTABLE" FOR FINITE $\cup$ COUNTABLY INFINITE

  "DENUMERABLE" FOR COUNTABLY INFINITE

THM: $\mathbb{Q}$ IS COUNTABLE

PF: FOR EACH $n \in \mathbb{N}$, LET $A_n = \left\{ \pm\frac{p}{q} \;\middle|\; p,q \in \mathbb{N},\ \gcd(p,q)=1,\ p+q=n \right\}$

SO $A_1 = \left\{\frac{0}{1}\right\}$, $A_2 = \left\{\frac{1}{1}, \frac{-1}{1}\right\}$, $A_3 = \left\{\frac{1}{2}, \frac{-1}{2}, \frac{2}{1}, \frac{-2}{1}\right\}$

$A_4 = \left\{\frac{1}{3}, \frac{-1}{3}, \frac{3}{1}, \frac{-3}{1}\right\}$, $A_5 = \left\{\frac{1}{4}, \frac{-1}{4}, \frac{2}{3}, \frac{-2}{3}, \frac{3}{2}, \frac{-3}{2}, \frac{4}{1}, \frac{-4}{1}\right\} \ldots$

OBSERVE THAT $\mathbb{Q} = \bigcup_{n=1}^{\infty} A_n$, $A_n \cap A_m = \emptyset$ IF $n \neq m$,
$|A_n| < \infty$ FOR EACH $n$.

WE CAN CONSTRUCT OUR BIJECTION AS

$$
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\
\updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\
\underbrace{\tfrac{0}{1}}_{A_1} & \multicolumn{2}{c}{\underbrace{\tfrac{1}{1} \quad \tfrac{-1}{1}}_{A_2}} & \multicolumn{4}{c}{\underbrace{\tfrac{1}{2}, \quad \tfrac{-1}{2}, \quad \tfrac{2}{1}, \quad \tfrac{-2}{1}}_{A_3}} & \multicolumn{2}{c}{\underbrace{\tfrac{1}{3}, \quad \cdots}_{A_4}}
\end{array}
$$

THM: IF $A \subseteq B$ AND $B$ IS COUNTABLE, THEN $A$ IS COUNTABLE, FINITE, OR EMPTY

THM: LET $A_1, A_2, A_3, \ldots A_n, \ldots$ EACH BE COUNTABLE.

- $A_1 \cup A_2 \cup \ldots \cup A_m$ IS COUNTABLE.
- $\bigcup_{n=1}^{\infty} A_n$ IS COUNTABLE.

Thm: $\mathbb{Q}$ is countable.

Pf: For each $n \in \mathbb{N}$, let $A_n = \{\pm\frac{p}{q} \mid p, q \in \mathbb{N}, \text{ and } p+q=n\}$

So $A_1 = \{\frac{0}{1}\}$, $A_2 = \{\frac{1}{1}, -\frac{1}{1}\}$, $A_3 = \{\frac{1}{2}, -\frac{1}{2}, \frac{2}{1}, -\frac{2}{1}\}$,

$A_4 = \{\frac{1}{3}, -\frac{1}{3}, \frac{3}{1}, -\frac{3}{1}\}$, $A_5 = \{\frac{1}{4}, -\frac{1}{4}, \frac{2}{3}, -\frac{2}{3}, \frac{3}{2}, -\frac{3}{2}, \frac{4}{1}, -\frac{4}{1}\}$ ...

Observe that $\mathbb{Q} = \bigcup_{n=1}^{\infty} A_n$, $A_n \cap A_m = \emptyset$ if $n \neq m$,

$|A_n| < \infty$ for each $n$.

We can construct our bijection as:

$$\underbrace{0}_{A_1}, \underbrace{1, -1}_{A_2}, \underbrace{\tfrac{1}{2}, -\tfrac{1}{2}, 2, -2}_{A_3}, \underbrace{\tfrac{1}{3}, -\tfrac{1}{3}, 3, -3}_{A_4}, \ldots$$

Thm 2: If $A \subseteq B$ and $B$ is countable, then $A$ is countable, finite, or empty.

Thm: If $A_1, A_2, A_3, \ldots, A_n, \ldots$ each is countable:

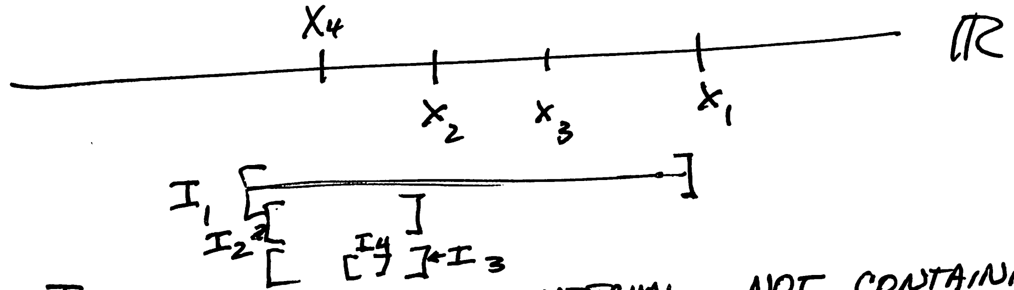
- $A_1 \cup A_2 \cup \ldots \cup A_m$ is countable.
- $\bigcup_{n=1}^{\infty} A_n$ is countable.

THM $\mathbb{R}$ IS UNCOUNTABLE

PF/ SPOZE THERE $\mathbb{R}$ IS COUNTABLE, IE

$$\mathbb{R} = \{x_1, x_2, x_3, \dots\}$$

NOW LET'S CONSTRUCT A NUMBER $x$ NOT ON THE LIST.




LET $I_1$ BE A CLOSED INTERVAL NOT CONTAINING $x_1$.

LET $I_2 \subset I_1$ BE A ~~[illegible]~~ CLOSED INTERVAL WITHIN $I_1$ NOT CONTAINING $x_2$

(CERTAINLY THERE IS ATLEAST ONE, SINCE $I_1$ CONTAINS TWO DISJOINT CLOSED INTERVALS, AND $x_2$ CAN'T BE IN BOTH)

IN GENERAL, FORM $I_n \subseteq I_{n+1}$ SO THAT $x_{n+1} \notin I_{n+1}$.

CONSIDER $\bigcap_{n=1}^{\infty} I_n$. IT IS NON-EMPTY BY THE NESTED INTERVAL ~~THEOREM~~ PROPERTY, BUT $x_k \notin \bigcap_{n=1}^{\infty} I_n$, SINCE $x_k \notin I_j$ FOR ALL $j \geq k$.

SO THERE MUST BE $x \in \mathbb{R}$ THAT CANT BE ONE OF THE $x_j$.

IE $|\mathbb{R}| \neq |\mathbb{N}|$. IN FACT $|\mathbb{R}| > |\mathbb{N}|$

# ANOTHER PROOF (CANTOR DIAGONALIZATION)

GEORG. CANTOR ~1891

THE INTERVAL $(0,1) = \{x \in \mathbb{R} \mid 0<x<1\}$ IS UNCOUNTABLE.

Pf/ ASSUME THERE IS $f: \mathbb{N} \to \mathbb{R}$ WHICH IS 1-1 AND ONTO.

FOR EACH $m \in \mathbb{N}$, $f(m)$ HAS AN INFINITE DECIMAL EXPANSION, IE

$$f(m) = 0.\, a_{m,1}\, a_{m,2}\, a_{m,3}\, a_{m,4} \cdots$$

WITH $a_{m,n} \in \{0,1,2,\dots,9\}$. FOR DEFINITENESS, ~~WE~~ ASSUME NO EXPANSION ENDS IN ALL 9s. THIS DOESN'T ACTUALLY MATTER, THOUGH.

~~NOW~~ SO WE HAVE

$$\begin{aligned}
1 &\leftrightarrow f(1) = .a_{11}\ a_{12}\ a_{13}\ a_{14}\ a_{15} \cdots \\
2 &\leftrightarrow f(2) = .a_{21}\ a_{22}\ a_{23}\ a_{24}\ a_{25} \cdots \\
3 &\leftrightarrow f(3) = .a_{31}\ a_{32}\ a_{33}\ a_{34} \cdots \\
4 &\leftrightarrow f(4) = .a_{41}\ a_{42}\ a_{43}\ a_{44} \cdots \\
&\quad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \ddots
\end{aligned}$$

~~WE A~~ EVERY REAL NUMBER SHOULD APPEAR ON THIS LIST. BUT IT DOESN'T.

LET $x = .b_1\, b_2\, b_3\, b_4 \cdots$

WHERE $b_n = \begin{cases} 4 & \text{IF } a_{nn} \neq 4 \\ 7 & \text{IF } a_{nn} = 4. \end{cases}$

$x$ CAN NOT APPEAR ON THE LIST. IF IT DID, IT WOULD BE AT SOME PLACE $k$ ON THE LIST, IE $f(k) = x$.

BUT THEN $b_k = a_{kk}$, IMPOSSBLE

DEF/ FOR A SET $A$, ITS **POWER SET** $\mathcal{P}(A)$ IS THE SET OF ALL SUBSETS OF $A$.

Ex: • LET $A = \{a, b, c\}$. THEN $\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$

• IF $|A| = n$, THEN $|\mathcal{P}(A)| = 2^n$

(IDEA OF PROOF: NUMBER THE ELEMENTS OF $A$ BY $1, 2, 3, \ldots, n$. ~~FOR~~ EACH SUBSET CORRESPONDS TO MAKING A CHOICE (IN, OUT) FOR EACH $a_i$.)

THM: (CANTOR'S THM) FOR ANY SET $A$, THERE IS NO SURJECTION $f: A \to \mathcal{P}(A)$

PF/ GIVEN A SET $A$, ~~WE CONSIDER~~ LET $f: A \to \mathcal{P}(A)$ BE ANY FUNCTION. WE SHOW $f$ IS NOT SURJECTIVE BY CONSTRUCTING $B \subseteq A$ WHICH IS NOT THE IMAGE OF $f(a)$ FOR ANY $a \in A$. LET $B = \{a \in A \mid a \notin f(a)\}$.

IF $f$ IS ONTO, THEN THERE IS AN $a'$ SO THAT $B = f(a')$.

IF $a' \in B$, WE CONTRADICT THE DEF. OF $B$.

IF $a' \notin B$, THEN BY DEF OF $B$, $a' \in B$.

$\boxtimes$

SO WE'VE SEEN THAT

$|\mathbb{N}| \neq |\mathbb{R}|$, IE THERE ARE "MORE" REALS THAN NATURALS,

COUNTABLE UNCOUNTABLE

BUT "THE SAME NUMBER" OF RATIONALS AS NATURALS.

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$|\mathbb{N}| = \aleph_0$ ("ALEPH-NOUGHT" OR "ALEPH-NULL")

$|\mathbb{R}| = \mathfrak{c}$

IT MAKES SENSE TO SAY $\aleph_0 < \mathfrak{c}$ (AS "CARDINAL NUMBERS")

IS THERE ANYTHING BETWEEN?

IT TURNS OUT THAT THAT THIS IS AN INDEPENDENT AXIOM. THAT IS, ~~THAT~~ IT IS CONSISTENT TO SAY "YES", AND ALSO TO SAY "NO".

THE CONTINUUM HYPOTHESIS

THERE ARE NO CARDINALS BETWEEN $\aleph_0$ AND $\mathfrak{c}$.

THAT IS, $|\mathbb{R}|$ IS THE "SMALLEST" UNCOUNTABLE SET.

NOTE THAT WE ~~DON'T PROVE~~ HAVEN'T SHOWN THAT $|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$ (BUT IT IS TRUE). ~~But~~ IDEA: REPRESENT $\mathcal{P}(\mathbb{N})$ AS AN INFINITE STRING OF 0 AND 1, WHICH GIVES AN INFINITE DECIMAL ...

NOTE ALSO THAT $|\mathcal{P}(\mathbb{R})| > |\mathbb{R}|$.