LAST TIME, TALKED ABOUT CONVERGENCE OF INFINITE SUMS (aka SERIES).

<u>DEF</u> : AN INFINITE SUM (OR SERIES)

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots$$

IS SAID TO CONVERGE TO $L$ IF THE SEQUENCE OF PARTIAL SUMS $\{S_m\}_{m=0}^{\infty}$ ~~WHICH~~ CONVERGES TO $L$ (i.e. $\lim_{n\to\infty} S_m = L$).

THE SEQUENCE OF PARTIAL SUMS IS GIVEN BY

$$S_m = \sum_{n=0}^{m} a_n = a_0 + a_1 + \cdots + a_m$$

IF $\lim_{n\to\infty} S_m$ DOES NOT EXIST, $\sum_{n=0}^{\infty} a_n$ <u>DIVERGES</u>

NOW CONSIDER THE IMPORTANT EXAMPLE OF THE <u>HARMONIC SERIES</u>

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

THIS <u>DIVERGES</u> (EASIEST PROOF USES THE INTEGRAL TEST. FIRST PROOF IN $14^{TH}$ CENT. BY <u>NICHOLAS ORESME</u> SIMILAR TO ONE BELOW)

NAME IS RELATED TO OVERTONES IN MUSIC, IE WAVELENGTHS OF STRING VIBRATIONS

$$\sum \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \cdots$$

THM: THE HARMONIC SERIES DIVERGES

(ORESME ~1350, LOST. REPROVEN ~1750 (MENGOLI)
~1780 (JOHANN BERNOULLI))

Pf/ LETS LOOK AT CERTAIN PARTIAL SUMS $H_k = \sum_{i=1}^{k} \frac{1}{i}$

$$H_1 = 1$$

$$H_2 = 1 + \frac{1}{2}$$

$$H_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right)$$
$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + 2\left(\frac{1}{2}\right)$$

$$H_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$$
$$> 1 + \frac{1}{2} + \frac{1}{2} + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + 3 \cdot \frac{1}{2}$$

$$H_{2^k} \geq 1 + \frac{k}{2}.$$

SINCE $\lim_{k \to \infty} H_{2^k}$ DIVERGES, IT IS IMPOSSIBLE FOR $\lim_{n \to \infty} H_n$ TO CONVERGE. $\boxtimes$

ANOTHER PROOF:

SUPPOSE $S = \sum_{n=1}^{\infty} \frac{1}{n}$. THEN

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots$$
$$= \left(1 + \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6}\right) + \left(\frac{1}{7} + \frac{1}{8}\right) + \cdots$$
$$> \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{6} + \frac{1}{6}\right) + \left(\frac{1}{8} + \frac{1}{8}\right) + \cdots$$
$$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = S.$$

SO $S > S$! $\Rightarrow\Leftarrow$

ORESME'S PROOF IS A SPECIAL CASE OF

## THE CAUCHY CONDENSATION TEST

SPOZE $\{b_n\}$ IS DECREASING WITH $b_n \geq 0$ FOR ALL $n$.

THEN $\sum_{n=0}^{\infty} b_n$ CONVERGES $\iff \sum_{n=0}^{\infty} 2^n b_{2^n} = b_1 + 2b_2 + 4b_4 + \cdots$ DOES.

Pf/ $\Rightarrow$ ALMOST IDENTICAL TO ORESME'S PROOF (USING CONTRA-POSITIVE)
IF $\sum 2^n b_{2^n}$ DIVERGES, SO DOES $\sum b_n$

$\Leftarrow$ ALSO SIMILAR, (SKIP IN CLASS)

SPOZE $\sum 2^n b_{2^n}$ CONVERGES.

THEN THE PARTIAL SUMS $t_k = b_1 + 2b_2 + 4b_4 + \cdots + 2^k b_{2^k}$
ARE BOUNDED, IE $\exists$ $B$ SO THAT $t_k \leq B$ FOR ALL $k \in \mathbb{N}$

WANT TO SHOW $\{S_m\}$ $b_1 + b_2 + b_3 + \cdots + b_m = S_m$
IS BOUNDED.

SO FIX $m$, AND ASSUME $k$ IS BIG ENOUGH
SO THAT $m \leq 2^{k+1} - 1$; SO $S_m \leq S_{2^{k+1}-1}$

AND

$$S_m < S_{2^{k+1}-1} = b_1 + (b_2 + b_3) + (b_4 + b_5 + b_6 + b_7) + \cdots + (b_{2^k} + b_{2^k+1} + \cdots b_{2^{k+1}-1})$$
$$\leq b_1 + (b_2 + b_2) + (b_4 + b_4 + b_4 + b_4) + \cdots + (b_{2^k} + b_{2^k} + \cdots b_{2^k})$$
$$= b_1 + 2b_2 + 4b_4 + \cdots + 2^k b_{2^k} = t_k$$

SO $S_m \leq t_k \leq B$, SO $\{S_m\}$ IS BOUNDED AND MONOTONE.
THUS IT CONVERGES.

**COROLLARY** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ CONVERGES $\iff p > 1$

(EASY PROOF BY INTEGRAL TEST. USE CAUCHY CONDENSATION ON HW #4)

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LET $\{a_n\}$ BE A BOUNDED SEQUENCE,

AND LET $\{y_n\}$ BE GIVEN BY

$$y_k = \sup\{a_k \mid k \geq n\}.$$

**DEF** $\limsup a_n = \lim y_n$ (IF $\{a_n\}$ IS NOT BOUNDED ABOVE $\limsup a_n = +\infty$)

"LIMIT SUPERIOR" OR $\overline{\lim}\, a_n$

$$\liminf a_n = \lim_{n\to\infty} \left(\inf\{a_k \mid k \geq n\}\right)$$

LIMIT INFERIOR $\underline{\lim}\, a_n$

IF $\{a_n\}$ NOT BDD BELOW $\liminf a_n = -\infty$

**FACT**

- $\liminf a_n \leq \limsup a_n$ FOR ANY SEQUENCE.
- $\liminf a_n = \limsup a_n \iff \lim_{n\to\infty} a_n$ EXISTS.

CONSIDER $\{a_n\} \left\{ (-1)^n \frac{n-1}{n} \right\}_{n=1}^{\infty} = 0, \frac{1}{2}, \frac{-2}{3}, \frac{3}{4}, \frac{-4}{5}, \frac{5}{6}, \ldots$

$$\limsup a_n = +1 \quad , \quad \liminf a_n = -1$$

**DEF:** LET $\{a_n\}$ BE A SEQUENCE

AND LET $n_1 < n_2 < n_3 < n_4 < \cdots$ BE A STRICTLY INCREASING SEQUENCE FROM $\mathbb{N} \cup \{0\}$

THEN THE SEQUENCE

$$\{a_{n_k}\}_{k=1}^{\infty} = a_{n_1}, a_{n_2}, a_{n_3}, \ldots$$

IS A <u>SUBSEQUENCE</u> OF $\{a_n\}$

OBSERVE THAT THE ORDER IS PRESERVED.

IN ★ WE HAVE TWO CONVERGENT SUBSEQUENCES EVEN THOUGH $\{a_n\}$ DIVERGES:

$$\{b_n\} = \{a_{2n}\}_{n=1}^{\infty} \to +1$$

$$\{c_n\} = \{a_{2n-1}\}_{n=1}^{\infty} \to -1$$

**THM:** IF $\{a_n\}$ CONVERGES TO $L$, ALL SUB SEQ'S OF $\{a_n\}$ ALSO CONVERGE TO $L$.