MAT 513

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2/14/22

RECALL: DEF OF LIMIT $\lim_{n\to\infty} a_n = L \iff \forall \varepsilon>0, \exists K_\varepsilon \in \mathbb{N}$ SO THAT $n > K_\varepsilon \Rightarrow |a_n - L| < \varepsilon$

ALSO • IF A SEQUENCE

a_n IS INCREASING AND BOUNDED, $\lim a_n = \sup\{a_n\}$

b_n IS DECREASING AND BOUNDED, $\lim b_n = \inf\{b_n\}$

AKA MONOTONE CONVERGENCE THEOREM

THM: IF A SEQUENCE IS MONOTONE AND BOUNDED, THEN IT CONVERGES.

Pf IS EASY.

ANOTHER USEFUL, EASY THEOREM:

THE SQUEEZE THEOREM

THM: SPOZE FOR $\{x_n\}, \{y_n\}, \{z_n\}$ WE HAVE

$x_n \leq y_n \leq z_n$ FOR ALL n SUFF. LARGE

IF $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = L$, THEN $\lim y_n = L$

Pf: NOTICE THAT THIS GENERALIZES THE MCT ABOVE

(TAKE IF $\{x_n\}$ INCREASING, TAKE $z_n = L = \lim \ldots$)

SO NOW TO PROVE.

FIX $\varepsilon > 0$.

WANT TO SHOW $y_n \to L$, IE $\exists K_y$ SO THAT $n > K_y \Rightarrow |y_n - L| < \varepsilon$.

BUT SINCE $x_n \to L$, HAVE K_x SO THAT $n > K_x \Rightarrow |x_n - L| < \varepsilon/2$

" " $z_n \to L$ " K_z SO $n > K_z \Rightarrow |z_n - L| < \varepsilon/2$

SO TAKE $K_y = \max(K_x, K_z)$.

THEN IF $n > K_y$, $|y_n - L| < |x_n - L| + |z_n - L| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

$\boxtimes$

NOW, LETS TURN (BRIEFLY) TO SPECIAL SEQUENCES CALLED SERIES. (OR INFINITE SUMS)

WE WANT TO ADD UP INFINITELY MANY NUMBERS,

EG.

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \sum_{n=0}^{\infty} \frac{1}{2^n}.$$

YOU PROBABLY KNOW THIS SUM IS 2,

OR MORE GENERALLY $\displaystyle\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ IF $|r| < 1$.

WHY IS THIS A SEQUENCE?

WHAT DOES $\sum_{n=0}^{\infty} a_n$ MEAN?

DEF GIVEN A SEQUENCE $\{a_n\}_{n=0}^{\infty}$,

FORM A NEW SEQUENCE $\{S_m\}_{m=0}^{\infty}$ (THE PARTIAL SUMS)

AS $$S_m = \sum_{n=0}^{m} a_n = a_0 + a_1 + a_2 + \cdots + a_m$$

THEN (DEF) $\sum_{n=0}^{\infty} a_n$ CONVERGES TO $L \iff \{S_m\} \to L$

OTHERWISE $\sum a_n$ DIVERGES

LET'S GO BACK TO GEOMETRIC SERIES EXAMPLE: $\left(\sum_{n=0}^{\infty} r^n\right)$

$$S_m = 1 + r + r^2 + r^3 + \cdots + r^m$$

$$rS_m = r + r^2 + r^3 + \cdots + r^m + r^{m+1}$$

SUBTRACT

$$(1-r)S_m = 1 - r^{m+1}$$

SO $$\boxed{S_m = \frac{1-r^{m+1}}{1-r}}$$

$$\lim_{m\to\infty} S_m = \frac{1}{1-r} \text{ IF } |r|<1 \quad \left(\text{BECAUSE } \lim_{m\to\infty} r^{m+1} \to 0\right)$$

ELSE DIVERGES ~~ELSE~~.

SO WE'VE PROVEN
$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \iff |r|<1$$

LETS TRY ANOTHER

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

$$\begin{aligned} S_m &= 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{m^2} \\ &< 1 + \frac{1}{2\cdot 1} + \frac{1}{3\cdot 2} + \cdots + \frac{1}{m(m-1)} \\ &= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{m-1} - \frac{1}{m}\right) \\ &= 1 + 1 + 0 + 0 + \cdots + 0 - \frac{1}{m} \\ &= 2 - 1/m. \end{aligned}$$

SO $\{S_m\}$ IS INCREASING, BOUNDED ABOVE BY 2.

$\therefore \sum_{n=1}^{\infty} \frac{1}{n^2}$ CONVERGES TO SOME NUMBER LESS THAN 2.

(IN FACT, THE SUM IS $\frac{\pi^2}{6}$)

WHAT ABOUT $\sum \frac{1}{n}$ (THE HARMONIC SERIES)?

THIS DIVERGES:

$$S_m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}$$

LOOKS LIKE IT SHOULD CONVERGE, BUT

$$S_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 2$$

$$S_8 > 2\tfrac{1}{2}$$

$$S_{2^k} > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots + 2^{k-1}\left(\frac{1}{2^k}\right) = 1 + \frac{k}{2}.$$

SO BIG AS YOU WANT.