LAST TIME

- $\mathbb{R}$ IS A COMPLETE ORDERED FIELD CONTAINING $\mathbb{Q}$.

- IN SOME SENSE, EVERYWHERE THERE IS A "HOLE" IN $\mathbb{Q}$, A NEW IRRATIONAL # IS INSERTED. BUT THERE ARE "MORE" IRRATIONALS THAN RATIONALS, EVEN THOUGH ~~THERE~~ BETWEEN EVERY RATIONAL THERE IS AN IRRATIONAL AND VICE VERSA (HW).

- WE USED THE AXIOM OF COMPLETENESS TO FILL IN THESE GAPS.

AXIOM OF COMPLETENESS: EVERY NONEMPTY SET OF REAL #S THAT IS BOUNDED ABOVE HAS A LEAST UPPER BOUND.

DEF: $A \subseteq \mathbb{R}$ IS BOUNDED ABOVE IF THERE EXISTS $b \in \mathbb{R}$ SO THAT $a \leq b$ FOR ALL $a \in A$. SUCH A b IS CALLED AN UPPER BOUND FOR THE SET A.

THE SET A IS BOUNDED BELOW IF THERE IS A LOWER BOUND l SO THAT $l \leq a$ FOR ALL $a \in A$.

DEF A REAL NUMBER s IS THE <u>LEAST UPPER BOUND</u> FOR $A \subseteq \mathbb{R}$ IF

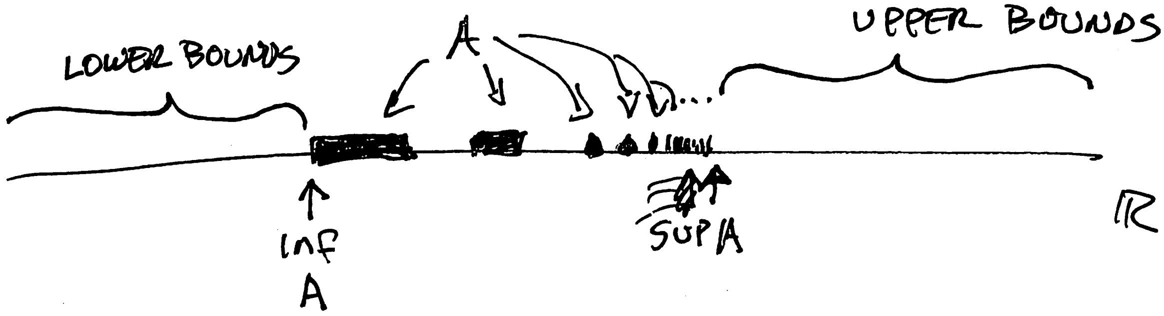
- s IS AN UPPER BOUND FOR A, AND
- ~~IF~~ b IS AN UPPER BOUND FOR A, THEN $s \leq b$.

LEAST UPPER BOUND IS AKA SUPREMUM OF A.
l.u.b.   sup.

SIMILARLY, WE CAN DEFINE THE <u>GREATEST LOWER BOUND</u> OF A,

$$\mathrm{glb}(A) = \inf(A)$$

INFIMUM.



~~IDEA~~ <u>EX</u> : $A = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\} = \{\frac{1}{n} \mid n \in \mathbb{N}\}$

$\inf A = 0$, $\sup A = 1$

$\sup A$ IS SIMILAR TO THE MAXIMUM, BUT NOT THE SAME. WHY NOT?

WHAT SHOULD DEF OF MAXIMUM BE?

HINT: $[0,2) = \{x \mid 0 \leq x < 2\}$ HAS NO MAX, BUT SUP IS 2.

HERE'S ANOTHER INTERPRETATION OF THE 2ND ~~PART~~ PART ~~OF~~ THE DEF FOR SUPREMUM:

LEMMA: LET $s \in \mathbb{R}$ BE AN UPPER BOUND FOR $A \subseteq \mathbb{R}$.

THEN $s = \sup A \iff$ FOR EVERY $\varepsilon > 0$, THERE IS SOME $a \in A$ SO THAT $s - \varepsilon < a$.

Pf/ $\Rightarrow$ SPOZE $s = \sup A$. ~~[illegible]~~ FIX $\varepsilon > 0$, AND BY THE DEFINITION $s - \varepsilon$ CAN NOT BE AN UPPER BOUND (SINCE $s - \varepsilon < s$); ~~[illegible]~~ HENCE THERE MUST BE $a \in A$ WITH $s - \varepsilon < a$.

$\Leftarrow$ SIMILARLY, ~~[illegible]~~ ASSUME s IS AN UPPER BOUND AND FOR EVERY $\varepsilon > 0$, ~~[illegible]~~ THERE IS $a \in A$ BIGGER THAN $s - \varepsilon$.

NOW SUPPOSE b IS ALSO AN UPPER BOUND FOR A, WITH $b \neq s$. THEN EITHER

$b < s$ (IN WHICH CASE THERE IS $a \in A$ WITH $b < a \leq s$, SINCE WE LET $\varepsilon = s - b > 0$ SO b NOT AN UPPER BD)

OR $b > s$ AS DESIRED.

$\boxtimes$

NOTE THAT AoC NOT THE CONGRESSWOMAN ONLY DISCUSSES UPPER BOUNDS, BUT IT IS EASY TO PROVE THAT IT IMPLIES THAT ANY SET BDD BELOW HAS AN INFIMUM.

ONE WAY: LET A BE BOUNDED BELOW

$B = \{ b \in \mathbb{R} \mid b \text{ is a lower bound for } A \}$

THEN $\sup B = \inf A$

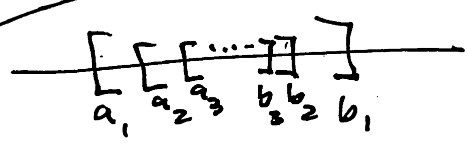
ANOTHER WAY: CONSIDER $C = \{ -a \mid a \in A \}$.

CONSEQUENCES OF COMPLETENESS:

- THE NESTED INTERVAL PROPERTY

FOR EACH $n \in \mathbb{N}$, SUPPOSE WE HAVE INTERVALS $I_n = [a_n, b_n]$ SO THAT $I_n \supseteq I_{n+1}$, THAT IS $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$

THEN $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.



PF/ CERTAINLY EACH I_n IS BOUNDED ABOVE & BELOW BY b_n & a_n

BY AoC, ~~THERE ARE NUMBERS $y = \inf\{b_n\}$, $x = \sup\{a_n\}$~~.

WE CAN LET $x = \sup\{a_n\}$. SINCE x IS AN UPPER BOUND FOR $\{a_n\}$, AND EACH b_n IS ALSO, ~~WE HAVE X~~

WE HAVE $a_n \leq x \leq b_n$ FOR EVERY n.

HENCE $\bigcap I_n$ CONTAINS x SO IS NONEMPTY.

ARCHIMEDEAN PROPERTY

(i) GIVEN ANY $x \in \mathbb{R}$, THERE IS $n \in \mathbb{N}$ SO THAT $n > x$

(ii) FOR ANY $y \in \mathbb{R}$ WITH $y > 0$, $\exists\, n \in \mathbb{N}$ SO THAT $\frac{1}{n} < y$.

NOTE THAT THIS DOES NOT CHARACTERIZE $\mathbb{R}$, SINCE IT IS ALSO TRUE OF $\mathbb{Q}$.

(i) JUST SAYS $\mathbb{N}$ IS NOT BOUNDED ABOVE.

Pf/ SPOZE IT WERE. THEN LET $\alpha = \sup \mathbb{N}$.

SO $\alpha - 1$ IS NOT AN UPPER BOUND, IE $\exists\, n \in \mathbb{N}$ WITH $n > \alpha - 1$.

BUT THEN $\alpha < n+1$, AND SINCE $n \in \mathbb{N}$, $n+1 \in \mathbb{N}$

SO α WAS NOT AN UPPER BOUND.

(ii) FOLLOWS QUICKLY FROM (i).

FOR $y > 0$, LET $z = 1/y$. APPLY (i) TO FIND $m \in \mathbb{N}$ WITH $z < m$. THEN $y = 1/z > 1/n > 0$.