1/24/2022

What is Analysis and why do I care?

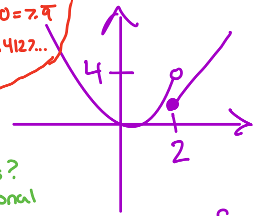
↳ calculus
change
series, sequences, $\boxed{\text{limits}}$ → Continuity of functions
Derivatives
Integrals

teaching Algebra I, II,
AP calc, 7th grade
5th grade

$\sqrt{2}$, $\frac{1}{3} = .333\ldots$ $3(\frac{1}{3}) =$
$1 = .999\overline{9}$
$2^3 = 8 = 8.00 = 7.\overline{9}$
$2^{\frac{1}{2}} = \sqrt{2} = 1.4127\ldots$
$(\sqrt{2})^3 = \sqrt{8}$
$3^{\sqrt{2}} = ?$

How do we do this? $\sqrt{2}$ is not a rational #. Same with $(\pi^{\sqrt{2}}) = (3.14159\ldots)^{1.4127\ldots}$

$$f(x) = \begin{cases} x^2 & x < 2 \\ 3x - 2 & x \geq 2 \end{cases}$$



This is a function – a lot of people don't think it is

$x = 2^{p/q}$ means $\sqrt[q]{2^p}$

$(x)^q = (2^{p/q})^q = 2^p$

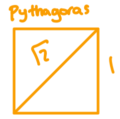
$x^q = 2^p$

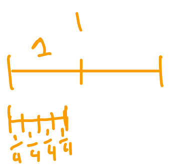
$x = (2^p)^{1/q}$

· There's a lot of subtlties. When x is an integer or rational, these things make sense.
But like given $x^2 = 2$, how do we know such a # exists? How do we know $\sqrt{2}$ exists?

· Can you prove $\sqrt{2}$ is irrational?

Number = length
Pythagoras





Start with (whole #'s) 1, 2, 3, 4, 5, ...
pretty quickly you'll have to think of ratios w/ these #'s
like 2/5 or 7/3 or whatever

Then later on came zero and negatives

$\ldots, -2, -1, 0, 1, 2, \ldots$ (Integers) $\mathbb{Z}$

Then we got rationals $\mathbb{Q}$

$3 + 7 = 10$ $\quad$ $3 - 7 = -4$

$3 \cdot 7 = \underbrace{7+7+7}_{3\text{ times}}$

$7^3 = \underbrace{7 \cdot 7 \cdot 7}_{3\text{ times}}$

So we have $(\mathbb{Z}, +, \cdot)$ and we can do stuff w/ them.

↓ extend

$(\mathbb{Q}, +, \cdot)$ Lets append the ability to divide

make sure we don't change the meaning of these things

$2 \div 3 = x$ means x solves $3x = 2$

include 0 and negatives: $3 + x = 2$

1st grader will say there's no solution

$x + y = y + x$

$x + (y + z) = (x + y) + z$

$x(y + z) = xy + xz$

What is 7^{-2}?

(Algebraic #'s, +, ·)

Solution to a polynomial equation → Does not cover the ratio or distance between these two

π Not algebraic (Transcendental)

gives ($\mathbb{R}$, +, ·) any "length"

(Real numbers)

Decimals! 0, 1, −2

rationals {1.25, 1.252525..., 1.33, 4/3

Add infinite things that don't repeat

irrational (can't write as a ratio of two whole #'s and can't repeat) → 1.0100100010001...

~ 49999 = ~ (+1500.00) have to do the same thing

Then we have all real #'s

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our goal is to understand this (and $f: \mathbb{R} \to \mathbb{R}$) function of the reals

hyperreal, infinitesimals

we lose some stuff + gain some stuff

$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \dots$

we lose the order (can't compare 2 complex #'s such as 2+i or 2−i)

and the algebraic operations agrees

Fundamental Idea: each "bigger" set of #'s agrees w/ the smaller one

often where you lose students in school math → your adding a new thing that builds on new stuff

Let's assume that you are "experts" with $\mathbb{Q} = \{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q > 0, \gcd(p,q) = 1 \}$

otherwise issue with $\frac{1}{2}$ + $\frac{-1}{-2}$

so $\frac{1}{2}$ and $\frac{2}{4}$ are same

- $\mathbb{Q}$ is an ordered field
  can compare everyone and put them in order
- A set is ordered means $\forall x, y \in S$ (set S), either $\underbrace{x < y, x = y, \text{ or } y < x}_{\text{Trichotomy Property}}$ (exclusive)

equivalent

Could also say $x \geq y$ or $y \geq x$

b/c if $x \geq y$ and $y \geq x$, then $x = y$

→ so if $x \geq y$ and $x \neq y$ then write $x > y$

But what if we have 3 things?

- For any x, y, z
  if $x \leq y$ and $y \leq z \Rightarrow x \leq z$

so include this in definition must have this so don't go in circles like a clock

x y z

- A field $\mathbb{F}$ is a set with two binary operations +, ·

so that $\forall x, y \in \mathbb{F}$ $\left.\begin{aligned} x + y &= y + x \\ x \cdot y &= y \cdot x \end{aligned}\right\}$ (commutative)

Basically a field is a set that has all of the properties

$\forall x, y, z \in \mathbb{F}$ $\left.\begin{aligned} (x+y)+z &= x+(y+z) \\ (x \cdot y)z &= x \cdot (y z) \end{aligned}\right\}$ (associative)

There are two elements 0, 1 $(0 \neq 1) \in \mathbb{F}$ (identity)

$\forall x \quad x + 0 = x$

$\qquad x \cdot 1 = x$

So $\mathbb{Z}_4$ can't be a field b/c 2 has no multiplicative inverse

$\forall x \in \mathbb{F} \; \exists -x \in \mathbb{F}$ so that $x + (-x) = 0$ } (inverses)

$\forall x \in \mathbb{F} \; \exists x^{-1} \in \mathbb{F}$ so that $x \cdot x^{-1} = 1$ } multiplicative inverse for everything except 0

$\forall x, y, z \in \mathbb{F}$ } (distributive)

$x \cdot (y + z) = x \cdot y + x \cdot z$

Ex of another field: $\{0, 1\}$

Addition: $0 + 0 = 0$ / even + even = even

$0 + 1 = 1$

$1 + 1 = 0$ / odd + odd = even

$1 + 0 = 1$ - odd + even = odd

This is $\mathbb{N}$ except

$\{\text{even}\}$ $\{\text{odd}\}$

$\Downarrow$ $\Downarrow$

0 1

Often written as $\mathbb{Z}_2$ (integers mod 2)

Multiplication: $0 \cdot 0 = 0$ - multiply two even = even

$0 \cdot 1 = 1 \cdot 0 = 0$ - multiply even and odd = even

$1 \cdot 1 = 0$ - multiply two odd's = odd

2 has no multiplicative inverse

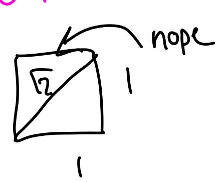
$\mathbb{Z}_3, \mathbb{Z}_5$ are also fields but not $\mathbb{Z}_4$ (something breaks when you observe the remainders)

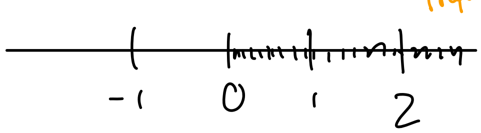
you might be thinking

- So $\mathbb{Q}$ is an ordered field. "Why can't we just work with $\mathbb{Q}$?" Stuff we want is missing. "$\mathbb{Q}$ is full of holes"

But it is dense in $\mathbb{R}$

i.e.





Archimedian Property?

If $r, s \in \mathbb{Q}$ with $r < s$, there is always $p \in \mathbb{Q}$ with $r < p < s$

always exists a rational # btwn them

i.e. all means