MAT513 Homework 10

Due Wednesday, May 3

Problems marked with a * are optional/extra credit. However, please at least consider them.

1. Let $f(x) = x^2$ on $[\frac{1}{2}, 3]$.

(a) Let $\mathcal{P}$ be the partition $\left\{\frac{1}{2}, 1, 2, 3\right\}$ and compute $L(f, \mathcal{P})$ and $U(f, \mathcal{P})$, where L and U are the lower and upper sums with respect to the partition.

(b) Let $\mathcal{Q} = \left\{\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3\right\}$ and compute $L(f, \mathcal{Q})$ and $U(f, \mathcal{Q})$.

2. Let f be continuous on $[a,b]$ and suppose that $f(x) \geq 0$ for all $x \in [a,b]$. Prove that if $L(f) = 0$, then $f(x) = 0$ for all $x \in [a,b]$.

***3.** Let $T\colon [0,1] \to [0,1]$ be the Thomae function and $h\colon [0,1] \to [0,1]$ be the Dirichlet function, both given below.

$$T(x) = \begin{cases} \frac{1}{q} & \text{if } x \in \mathbb{Q} \text{ with } x = \frac{p}{q} \text{ in least terms, } q > 0 \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases} \qquad h(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

The Thomae function is integrable (the integral is zero), but the Dirichlet function is not. Show that the composition of two integrable functions need not be integrable by finding an integrable function $g\colon [0,1] \to [0,1]$ so that $g(T(x)) = h(x)$.

4. Prove the **Mean Value Theorem for Integrals**: If f is continuous on $[a,b]$, then there exists $c \in (a,b)$ for which

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

(This value $f(c)$ is called the **average value of f** on the interval $[a,b]$.)

5. Assume that functions $u(x)$ and $v(x)$ have continuous derivatives on $[a,b]$. Derive the formula for **integration by parts**:

$$\int_a^b u(t)v'(t)\, dt = \Big(u(b)v(b) - u(a)v(a)\Big) - \int_a^b v(t)u'(t)\, dt.$$

***6.** Recall again the definition of the middle-thirds Cantor Set $\mathcal{C}$. By analogy with the Dirichlet function, define

$$\chi_{\mathcal{C}}(x) = \begin{cases} 1 & \text{if } x \in \mathcal{C} \\ 0 & \text{if } x \notin \mathcal{C} \end{cases}$$

This function $\chi_{\mathcal{C}}$ is discontinuous at each point of $\mathcal{C}$, and continuous at every point in the complement of $\mathcal{C}$. Thus, $\chi_{\mathcal{C}}$ is discontinuous on an uncountably infinite set.

Show that $\chi_{\mathcal{C}}$ is integrable on $[0,1]$.

7. Write an explanation of the intuitive heuristic behind the second part of the Fundamental Theorem of Calculus: Let $G(x)$ be the function that measures the area under the graph of g from a to x. Then the derivative of G at x is the height $y = g(x)$, since the approximate change from x to $x+h$ is essentially the area of the rectangle with base h and height y.

You might want to include a relevant picture, and relate this to the explicit statement of the relevant part of the FTC. Give your explanation so that it can be understood by a beginning calculus student.