MAT513 Homework 5

Due Wednesday, March 22

Problems marked with a * are optional/extra credit. However, please at least consider them.

1. Let K and L be compact subsets of $\mathbb{R}$. We can define a distance between K and L as

$$d(K,L) = \inf_{x\in K,\ y\in L} \{\, |x-y| \,\}.$$

(a) Show that if K and L are disjoint compact sets, then $d(K,L) > 0$.

(b) Give an example of disjoint closed sets A and B for which $d(A,B) = 0$.

2. For a function $f\colon A \to \mathbb{R}$ with $c \in \bar{A}$, recall that $\lim\limits_{x\to c} = L$ means that for every $\varepsilon > 0$ there exists $\delta > 0$ so that $|f(x) - L| < \varepsilon$ whenever $0 < |x-c| < \delta$ and $x \in A$.
Let $\lceil x \rceil$ denote the smallest integer greater than or equal to x (for example, $\lceil 0.5 \rceil = 1 = \lceil 1 \rceil$).

(a) Suppose we take $\varepsilon = 1$. What is the largest value of δ we can use in the definition of $\lim\limits_{x\to\pi} \lceil x/2 \rceil$?

(b) Suppose we take $\varepsilon = .01$. What is the largest value of δ we can use in the definition of $\lim\limits_{x\to\pi} \lceil x/2 \rceil$?

(c) Write a proof, using the definition of limit above, that $\lim\limits_{x\to\pi} \lceil x/2 \rceil = 2$.

(d) Consider $g(x) = \dfrac{1}{\lceil x \rceil}$. A student makes the (false) claim that $\lim\limits_{x\to 4} g(x) = \dfrac{1}{4}$. Give an explanation of why this cannot be true by exhibiting the largest ε for which there is no δ that satisfies the definition.

3. We can extend the definition of limit to include limits which are infinite. Specifically, for $f\colon A \to \mathbb{R}$, we replace the arbitrarily small $\varepsilon > 0$ with an arbitrarily large $M > 0$ (where $c \in \bar{A}$ as usual). Specifically, we say $\lim\limits_{x\to c} = +\infty$ if, for every $M > 0$ there exists $\delta > 0$ so that for all $x \in A$, having $0 < |x-c| < \delta$ ensures that $f(x) > M$.

(a) Using this definition, prove that $\lim\limits_{x\to 0} \dfrac{1}{x^2} = +\infty$.

(b) Construct an analogous definition for the statement $\lim\limits_{x\to+\infty} f(x) = L$, and use it to write a proof that $\lim\limits_{x\to+\infty} \dfrac{1}{x} = 0$.

***4**. Recall the definition of Thomae's function:

$$T(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{q} & \text{if } x = \frac{p}{q}, \text{ with } p \in \mathbb{Z},\ q \in \mathbb{N} \text{ and } \gcd(p,q) = 1 \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Show that $\lim\limits_{x\to 1} T(x) = 0$.

Hint: for each $\delta > 0$, consider the sets $A_\varepsilon = \{\, x \in \mathbb{R} \mid T(x) > \varepsilon \,\}$. Argue that for every fixed $\varepsilon > 0$, every point in A_ε is isolated.

5. Here are several invented definitions which are variations on the definition of continuity. In each case, if you give an example you must justify that it meets the stated criteria.

(a) A function $f\colon \mathbb{R} \to \mathbb{R}$ is **onetinuous** at c if for every $\varepsilon > 0$, we have $|f(x) - f(c)| < \varepsilon$ whenever $|x - c| < 1$.
Give an example of a function g that is onetinuous on all of $\mathbb{R}$, and another function h that is continuous at every $c \in \mathbb{R}$, onetinuous at $c = 0$, but not onetinuous at $c = 2$, or explain why no such function can exist.

(b) A function $f\colon \mathbb{R} \to \mathbb{R}$ is **equaltinuous** at c if for every $\varepsilon > 0$, whenever $|x - c| < \varepsilon$ we also have $|f(x) - f(c)| < \varepsilon$.
Give an example of a function f which is not onetinuous at any $c \in \mathbb{R}$, but is equaltinuous at every $c \in \mathbb{R}$, or explain why no such function can exist.

(c) A function $f\colon \mathbb{R} \to \mathbb{R}$ is **lesstinuous** at $c \in \mathbb{R}$ if for every $\varepsilon > 0$, there is a δ with $0 < \delta < \varepsilon$ so that $|f(x) - f(c)| < \varepsilon$ whenever $|x - c| < \delta$.
Find a function f which is lesstinuous on all of $\mathbb{R}$ but is nowhwere equaltinuous, or explain why no such function can exist.

(d) Is every lesstinuous function continuous? Is every continuous function lesstinuous? Explain.

6. Let A and B be subsets of $\mathbb{R}$, with $f\colon A \to B$ and $g\colon B \to \mathbb{R}$. Prove that if f is continuous at $c \in A$ and g is continuous at $f(c) \in B$, then $g \circ f$ is continuous at c.
You may use this using any of the characterizations of continuity given in Theorem 4.3.2.

7. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that the set $K = \{\, x \mid f(x) = 0 \,\}$ is a closed set.

***8**. Let $f\colon \mathbb{R} \to \mathbb{R}$ satisfy the property that $f(x+y) = f(x) + f(y)$ for every real number x and y. (Such a function is called an **additive homomorphism**.)

(a) Show that $f(0) = 0$ and $f(-x) = -f(x)$ for every $x \in \mathbb{R}$.

(b) Let $k = f(1)$, and show that $f(n) = kn$ for all $n \in \mathbb{N}$ (and hence, by the previous part, for all $n \in \mathbb{Z}$). Then show $f(r) = kr$ for all $r \in \mathbb{Q}$.

(c) Finally, show that if f is continuous at $x = 0$, then f is continuous at every $x \in \mathbb{R}$. Consequently, any additive homomorphism of $\mathbb{R}$ which is continuous at one point is a linear function of the form $f(x) = kx$.

9. A student says that there is no reason we need to define continutity in terms of limits, and that we can just say that *a function is continuous (on an interval) if we can draw the graph from start to finish without ever picking up our pencil (or crayon).*
Write a paragraph or more responding to the student's claim. Write your answer in such a way that it can be understood by a high-school student.