MAT513 Homework 2

Due Wednesday, February 15

1. Let $I = \{x \mid 0 < x < 1\}$ be the open unit interval $(0,1)$, and let S be the open unit square, that is, $S = \{(x,y) \mid 0 < x < 1 \text{ and } 0 < y < 1\} = (0,1) \times (0,1)$.

(a) Find an injective function (that is, a one-to-one function) $f\colon I \to S$. This should be *very easy*: f does not need to be surjective (onto).

(b) Use the fact that every real number x has a decimal expansion to produce an injective function $g\colon S \to I$. Is your function g a surjection (onto)?
It might be helpful to remember that every real number which has a "terminating" decimal expansion (such as 0.25) can also be written as an infinite decimal (e.g., $0.24999\ldots$).

As a consequence of the Schröder-Bernstein Theorem (which says that if there are injective functions $f\colon A \to B$ and $g\colon B \to A$, then there is a bijective function $h\colon A \to B$), this shows that the unit interval and the unit square have the same cardinality.

2. A real number $x \in \mathbb{R}$ is called **algebraic** if there are integers $a_0, a_1, a_2, \ldots, a_n$ so that

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0,$$

that is, $x \in \mathbb{R}$ is algebraic if it is a root of a polynomial with integer coefficients. Real numbers which are not algebraic are called **transcendental** numbers.

(a) Show that $\sqrt{2}$ and $\sqrt{3} + \sqrt{2}$ are algebraic.

(b) Fix $n \in \mathbb{N}$ and let A_n be set of algebraic numbers which are roots of polynomials of degree n. Show that each A_n is a countable set. (Hint: the Fundamental Theorem of Algebra is relevant here.)

(c) Prove that the set of algebraic numbers is a countable set.

(d) What is the cardinality of the set of transcendental numbers?

3. What happens if we reverse the order of the quantifiers in the definition of convergence? That is, consider the following invented definition:

A sequence $\{x_n\}$ **verconges** *to L if there exists $\varepsilon > 0$ such that for all $M \in \mathbb{N}$, we have $|x_n - L| < \varepsilon$ for all $n \in \mathbb{N}$ with $n \geq M$.*

Give an example of a sequence which is vercongent. Is there an example of a vercongent sequence which diverges (in the ordinary sense)? Can a sequence verconge to L and also to K with $L \neq K$?
Explain in ordinary English what this definition is describing.

4. Prove or disprove (by giving a counterexample) each of the following:

(a) If $\{s_n\}$ converges to S, then $\{|s_n|\} \to |S|$.

(b) If $\{|s_n|\}$ converges, then $\{s_n\}$ is convergent.

(c) $\lim s_n = 0$ if and only if $\lim |s_n| = 0$.

5. Using the definition of convergence of a sequence, prove that each of the following sequences converge to the given limit.

(a) $\dfrac{2n+1}{5n+4} \longrightarrow \dfrac{2}{5}$.

(b) $\dfrac{2n^2}{n^3+3} \longrightarrow 0$.

(c) $\dfrac{\sin(n^2)}{\sqrt{n}} \longrightarrow 0$.

6. Prove Theorem 2.2.7 (p. 46): *The limit of a sequence, when it exists, must be unique.* To do so, suppose that the sequence $\{a_n\} \to L$ and also $\{a_n\} \to K$. Then argue that $M = K$.

7. Let $P(x)$ be some property that x has.

- A sequence $\{a_n\}$ is said to **eventually** have property P if there exists $M \in \mathbb{N}$ such that $P(a_n)$ holds for all $n \geq M$.
- A sequence $\{a_n\}$ is said to **frequently** have property P if for every $M \in \mathbb{N}$, there exists $n \geq M$ for which $P(a_n)$ holds.

(a) Does the sequence $\{(-1)^n\}$ eventually take on the value 1? Does it frequently take on the value 1?

(b) Which is stronger? That is, does eventually always imply frequently? Does frequently always imply eventually? Give a proof or counterexample.

(c) Rephrase the usual definition of convergence of a sequence using either eventually or frequently (one works, the other doesn't. Which is it?)

8. (**The Squeeze Theorem**) Suppose that $x_n \to L$ and $z_n \to L$. Prove that if $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$, then $\{y_n\}$ converges to L.

9. Are each of the following possible? If so, give an example. If not, give a proof showing why it is impossible.

(a) The sequence $\{a_n\}$ is divergent and $\{b_n\}$ is convergent, while $\{a_n b_n\}$ converges.

(b) The sequence $\{a_n\}$ is unbounded and $\{b_n\}$ is convergent, but $\{a_n - b_n\}$ is bounded.

10. Consider Zeno's paradox of Achilles and the Tortoise; a version[1] is given below. (Or watch a video version on YouTube.)

The Tortoise challenged Achilles to a race, claiming that he would win as long as Achilles gave him a small head start. Achilles laughed at this, for of course he was a mighty warrior and swift of foot, whereas the Tortoise was heavy and slow.

"How big a head start do you need?" he asked the Tortoise with a smile.

"Ten meters," the latter replied.

Achilles laughed louder than ever. "You will surely lose, my friend, in that case," he told the Tortoise, "but let us race, if you wish it."

[1]from Smith, B. Sidney. "Zeno's Paradox of the Tortoise and Achilles", Platonic Realms Interactive Mathematics Encyclopedia. http://platonicrealms.com

"On the contrary," said the Tortoise, "I will win, and I can prove it to you by a simple argument."

"Go on then," Achilles replied, with less confidence than he felt before. He knew he was the superior athlete, but he also knew the Tortoise had the sharper wits, and he had lost many a bewildering argument with him before this.

"Suppose," began the Tortoise, "that you give me a 10-meter head start. Would you say that you could cover that 10 meters between us very quickly?"

"Very quickly," Achilles affirmed.

"And in that time, how far should I have gone, do you think?"

"Perhaps a meter— no more," said Achilles after a moment's thought.

"Very well," replied the Tortoise, "so now there is a meter between us. And you would catch up that distance very quickly?"

"Very quickly indeed!"

"And yet, in that time I shall have gone a little way farther, so that now you must catch that distance up, yes?"

"Ye-es," said Achilles slowly.

"And while you are doing so, I shall have gone a little way farther, so that you must then catch up the new distance," the Tortoise continued smoothly.

Achilles said nothing.

"And so you see, in each moment you must be catching up the distance between us, and yet I— at the same time— will be adding a new distance, however small, for you to catch up again."

"Indeed, it must be so," said Achilles wearily.

"And so you can never catch up," the Tortoise concluded sympathetically.

"You are right, as always," said Achilles sadly— and conceded the race.

Write a paragraph or so explaining how the idea of a convergent sequence resolves the apparent paradox. Suppose $A(t)$ represents the position of Achilles at time t, and $T(t)$ represents that of the Tortoise. From the above story, Achilles moves ten times as fast as the Tortoise. What can you say about the sequence of times $\{t_n\}$ given by $A(t_n) = T(t_{n-1})$, where $t_0 = 0$, $A(0) = 0$, $T(0) = 10$?

Not to hand in: Consider the paradox called Thomson's Lamp, devised in 1954 by James F. Thomson.

There is a lamp which has a toggle switch: Hitting the switch once turns the lamp on and hitting it again turns the lamp off.

Suppose you do the following: Turn the lamp on, then wait 1 minute. Turn the lamp off, wait 30 seconds. Then turn the lamp on and wait 15 seconds. Continue in this way: after a time period exactly half of the previous one, hit the switch.

At the end of two minutes, is the lamp on or off?

Compare this to the Ross-Littlewood paradox, where an infinite number of balls are added and removed from a vase (and I won't describe here, but you can look it up). Infinity is tricksy, even tricksier than hobbitses!