MAT513 Homework 1
Due Wednesday, February 8

1. Adapt the proof from class or the text (or give another) to prove that $\sqrt{3}$ is irrational. How does it need to be adjusted to prove that $\sqrt{6}$ is irrational? (Give the adjusted proof.) What about $\sqrt{12}$?

2. Let A^c denote the complement of the set A; since this is analysis, we may assume A consists of real numbers, so $A^c = \mathbb{R} \setminus A$.

(a) Use induction to show that

$$(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap A_3^c \cdots A_n^c.$$

You can assume the results of Ex. 1.2.5 in the text.

(b) Show that this *does not* generalize to an infinte collection of sets. That is, show that

$$\left(\bigcup_{i=1}^{\infty} A_i \right)^c \neq \bigcap_{i=1}^{\infty} A_i^c$$

by finding an example of sets $B_1, B_2, B_3, \ldots$ for which $\bigcap_{i=1}^{n} B_i \neq \emptyset$ for all $n \in \mathbb{N}$, but $\bigcap_{i=1}^{\infty} B_i = \emptyset$. (The relation with the first part is taking $A_i^c = B_i$.)

3. Use induction to prove that $5^{2n} - 1$ is divisible by 8 for all $n \in \mathbb{N}$.
(Hint: observe that $5^{2k+2} - 1 = 5^{2k+2} - 5^2 + 5^2 - 1$.)

4. We discussed the **Cut Property** in class, which says the following:

Let A and B be nonempty, disjoint sets of real numbers with $A \cup B = \mathbb{R}$ and so that $a < b$ for every $a \in A$ and every $b \in B$. For such sets A and B, the Cut Property *states that there exists a $c \in \mathbb{R}$ so that $a \leq c$ for every $a \in A$ and also $b \geq c$ for every $b \in B$. (Such a point c is called a* cut*).*

The Axiom of Completeness and the Cut Property are equivalent statements: if we take one of them as an axiom for $\mathbb{R}$, the other becomes a theorem. See also the discussion of Dedekind Cuts in §8.6 of the text.

(a) Give an example that shows that the Cut Property does not hold if we replace $\mathbb{R}$ by $\mathbb{Q}$. That is, find two disjoint, nonempty sets of rational numbers A and B so that $A \cup B = \mathbb{Q}$ and with each element of A strictly less than every element of B, but with A and B such that there is no rational number $r \in \mathbb{Q}$ satisfying $a \leq r$ for all $a \in A$ and also $r \leq b$ for all $b \in B$.

(b) Use the Axiom of Completeness to prove the cut property.

(c) Show that if we assume the Cut Property, the Axiom of Completeness follows from it.

5. Since $\mathbb{Q}$ is a field, we know that the sum and product of two rational numbers is also rational. Prove that if t is irrational and $a \in \mathbb{Q}$ with $a \neq 0$, then $a + t$ and at are both also irrational.

6. Prove that given any two real numbers a and b with $a < b$, there is an irrational number t for which $a < t < b$.
Hint: consider the numbers $a - \sqrt{2}$ and $b - \sqrt{2}$ and use the previous exercise.

7. Let $\mathbb{F}$ be the set of all rational functions, that is, the set

$$\mathbb{F} = \left\{ f(x) \ \middle|\ f(x) = \frac{p(x)}{q(x)} \text{ where } p(x) \text{ and } q(x) \text{ are polynomials} \right\}$$

Using the usual rules for addition and multiplication of rational functions, $\mathbb{F}$ can easily be shown to be a field.

Given a rational function $f \in \mathbb{F}$, we can define f to be *positive* whenever the leading coefficients (the coefficient of the highest power) of the numerator and denominator of f have the same sign. For example, the functions $\dfrac{3x^2+4x-1}{7x^5+5}$ and $\dfrac{2-5x^5}{17-2x^2}$ are both positive, but $\dfrac{1-2x^2}{x^4-15}$ is not.

We can define an ordering on $\mathbb{F}$ by saying that $f > g$ exactly when $f - g$ is positive in the above sense. This means $\mathbb{F}$ is an ordered field. Observe that we can view $\mathbb{N}$ as a subset of F by viewing $n \in \mathbb{N}$ as the constant function $f(x) = n$.

(a) Show by giving explicit counterexamples, that the field $\mathbb{F}$ does not have the Archimedian property. That is, find $f \in \mathbb{F}$ so that $f > n$ for all $n \in \mathbb{N}$, and also, find $g \in \mathbb{F}$ for which $0 < g < 1/n$ for all $n \in \mathbb{N}$.

(b) Show that $\mathbb{F}$ doesn't satisfy the completeness axiom by finding a subset $B \subset \mathbb{F}$ which is bounded above, but has no least upper bound. Justify your answer fully.

8. Write a paragraph or two responding to the following statement: "Because of the density of $\mathbb{Q}$ in $\mathbb{R}$, every measurement corresponding to a real number can be approximated by a rational number to within the precision of any device we can use measure it with. Thus, in science or engineering, it suffices to work only with real numbers that are fractions (or finite decimals, if you prefer)."