MAT 513 Final Exam

May 11, 2017

Name: ______________________ **ID:** __________________

Question:	1	2	3*	4*	5*	6*	7*	8*	Total
Points:	15	12	10	10	10	10	10	10	77
Score:									

There are 8 problems in this exam. Make sure that you have them all.

Do all of your work in this exam booklet, and cross out any work that the grader should ignore. You may use the backs of pages, but indicate what is where if you expect someone to look at it. **Books, calculators, extra papers, and discussions with friends are not permitted.** If you have statements of theorems or other information permanently tatooed somewhere on your body, you may refer to these (provided that the tatoo is located where it will be socially acceptable for you to refer to it in a classroom setting.)

Points will be taken off for writing mathematically false statements, even if the rest of the problem is correct.

You have 2.5 hours, more or less, to complete this exam.
If you finish early, you may find it useful use this time to review your answers on this exam (as well as all the choices you have made in your life up to this point.)

$\boxed{\text{15 points}}$ 1. (a) Suppose that for each $n \in \mathbb{N}$ we have $f_n \colon A \to \mathbb{R}$. Define what it means to say that the sequence f_n **converges uniformly on** $\boldsymbol{A}$.

(b) Suppose $f \colon A \to \mathbb{R}$. Define what it means for f to be **differentiable on** $\boldsymbol{A}$.

(c) State the **Fundamental Theorem of Calculus**.

$\boxed{\text{12 points}}$ 2. For each of the following, either provide an example (proof not needed) or a brief explanation of why no such object exists.

(a) A bounded set which contains its infimum but does not contain its supremum.

(b) A closed set which is not compact.

(c) A function $f\colon [0,1] \to \mathbb{R}$ which is differentiable but not integrable.

(d) A continuous function $f\colon [0,1] \to \mathbb{R}$ which is not uniformly continuous on $[0,1]$.

Do **any five** of the six questions 3–8.

10 points 3. Suppose that $a_n \geq 0$ and $\sum_{n=0}^{\infty} a_n$ converges. Prove that for every $\epsilon > 0$, there is a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ for which $\sum_{j=1}^{\infty} a_{n_j} < \epsilon$.

Do **any five** of the six questions 3–8.

10 points 4. Let $f\colon \mathbb{R} \to \mathbb{R}$ be differentiable.

(a) Show that if f and f' are both strictly increasing functions, then f is unbounded. Hint: the Mean Value Theorem is probably relevant.

(b) Give an example of a **bounded** differentiable function $g\colon \mathbb{R} \to \mathbb{R}$ where g is strictly increasing, but g' is not (or prove no such function g can exist).

Do **any five** of the six questions 3–8.

10 points 5. (a) Use the ϵ-δ definition to show that $f(x) = x^2$ is continuous at every $c \in [0, 3]$.

(b) Is f uniformly continuous on $(0, 3)$? Fully justify your answer.

Do **any five** of the six questions 3–8.

10 points 6. Suppose $f: \mathbb{R} \to \mathbb{R}$ is a nondecreasing function. Prove that at each point $c \in \mathbb{R}$, f is either continuous or has a jump discontinuity. (That is, show that no monotone function can have an essential or removable discontinuity at any point in its domain).

Do **any five** of the six questions 3–8.

10 points 7. (a) Let f be continuous on $[a, b]$ with $f(x) \geq 0$ for all $x \in [a, b]$. Suppose that there exists $c \in (a, b)$ for which $f(c) > 0$. Prove that $\int_a^b f(x)\, dx > 0$.

(b) Suppose f is nonnegative and integrable on $[a, b]$, and there exists $c \in (a, b)$ with $f(c) > 0$. Must it be true that $\int_a^b f(x)\, dx > 0$?
If so, give a proof; if not, give a counterexample.

Do **any five** of the six questions 3–8.

10 points 8. (a) Derive the Taylor series for $\ln(1+x)$. You may either derive it directly or via manipulation of another well-known series (e.g. the geometric series). For what x does the series converge?

(b) Use the first two nonzero terms of the series to estimate $\ln(3/2)$.

(c) Give an bound for the error in your answer to the previous part (and justify this bound).

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