MAT 513 Solutions to Final Exam

15 pts 1. (a) Suppose that for each $n \in \mathbb{N}$ we have $f_n \colon A \to \mathbb{R}$. Define what it means to say that the sequence f_n **converges uniformly on A**.

Solution: A sequence of functions $\{f_n\}$ converges uniformly on A to a function f if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ so that we have $|f_n(x) - f(x)| < \epsilon$ for all $x \in A$ and all $n \geq N$.

(b) Suppose $f \colon A \to \mathbb{R}$. Define what it means for f to be **differentiable on A**.

Solution: Given $c \in A$, the derivative at c is defined as $f'(c) = \lim_{x \to c} \dfrac{f(x) - f(c)}{x - c}$ whenever this limit exists. The function f is differentiable on A if $f'(c)$ exists for all $c \in A$.

(c) State the **Fundamental Theorem of Calculus**.

Solution: The fundamental theorem of calculus has two parts:

- Let $f \colon [a, b] \to \mathbb{R}$ be integrable, and suppose $F \colon [a, b] \to \mathbb{R}$ satisfies $F'(x) = f(x)$ for all $x \in [a, b]$. Then
$$\int_a^b f(x)\, dx = F(b) - F(a).$$
- Let $f \colon [a, b] \to \mathbb{R}$ be integrable and define $G(x) = \displaystyle\int_a^b g(t)\, dt$ for every $x \in [a, b]$. Then G is continuous on $[a, b]$, and if g is continuous at $c \in [a, b]$, then G is differentiable at c with $G'(c) = g(c)$.

12 pts 2. For each of the following, either provide an example (proof not needed) or a brief explanation of why no such object exists.

(a) A bounded set which contains its infimum but does not contain its supremum.

Solution: The half-open interval $[0, 1)$ does the job, as does a sequence like $\left\{1 - \frac{1}{n}\right\}$.

(b) A closed set which is not compact.

Solution: The interval $[0, \infty)$ works. Or you could use $\mathbb{N}$ or $\mathbb{Z}$ or plenty of other things.

(c) A function $f \colon [0, 1] \to \mathbb{R}$ which is differentiable but not integrable.

Solution: This is impossible, since every differentiable function is continuous, and every continuous function is integrable.

(d) A continuous function $f \colon [0, 1] \to \mathbb{R}$ which is not uniformly continuous on $[0, 1]$.

Solution: Again, impossible. If f is continuous on a compact set, it is uniformly continuous.

10 pts 3. Suppose that $a_n \geq 0$ and $\sum_{n=0}^{\infty} a_n$ converges. Prove that for every $\epsilon > 0$, there is a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ for which $\sum_{j=1}^{\infty} a_{n_j} < \epsilon$.

Solution: Let $\sum_{n=0}^{\infty} a_n = L$. This means that the sequence of partial sums $s_k = \sum_{n=0}^{k} a_n$ converges. That is, for every $\epsilon > 0$, there is an N so that $|L - s_k| < \epsilon$ for all $k \geq N$. But

$$|L - s_k| = \left| L - \sum_{n=0}^{k} a_n \right| = \left| \sum_{n=0}^{\infty} a_n - \sum_{n=0}^{k} a_n \right| = \left| \sum_{n=k+1}^{\infty} a_n \right|; \quad \text{in particular,} \quad |L - s_N| < \epsilon.$$

Thus, if we let $n_j = N + j$, that is, set $b_1 = a_{N+1}$, $b_2 = a_{N+2}$, etc., we will ensure that $|\sum a_{n_j}| = |\sum b_j| < \epsilon$, as desired.

10 pts 4. Let $f\colon \mathbb{R} \to \mathbb{R}$ be differentiable.

(a) Show that if f and f' are both strictly increasing functions, then f is unbounded. Hint: the Mean Value Theorem is probably relevant.

Solution: We can apply the Mean Value theorem to f to see that for some $c \in [0, 1]$, we have $f'(c) = f(0) - f(1)$. Since f is increasing, we have $f'(c) > 0$. Also, since f' is increasing, for all $x \geq c$, we have $f'(x) > f'(c) > 0$.

Denote $f'(c) = m$ and $f(c) = a$, and let $g(x)$ denote the line $g(x) = a + m(x - c)$. For all $x > c$, we have $f(x) > g(x)$, since $f(c) = g(c)$ and $f'(x) > g'(x)$ for $x > c$. Since $g(x)$ is unbounded, so is $f(x)$.

(b) Give an example of a **bounded** differentiable function $g\colon \mathbb{R} \to \mathbb{R}$ where g is strictly increasing, but g' is not (or prove no such function g can exist).

Solution: Several examples are possible, of course. But here is one:

Let $g(x) = \arctan(x)$. This function is an increasing function, but $g'(x) = 1/(1 + x^2)$ is increasing for $x < 0$ and decreasing for $x > 0$. Here we have a bounded function, since $-\pi/2 < \arctan(x) < \pi/2$ for all x.

Note that if $g'(x)$ is strictly increasing, $g(x)$ must be unbounded. But you weren't asked about that.

10 pts 5. (a) Use the ϵ-δ definition to show that $f(x) = x^2$ is continuous at every $c \in [0, 3]$.

Solution: Let $\epsilon > 0$ be arbitrary. We need to show that there is a δ so that whenever $0 < |x - c| < \delta$, we have $|x^2 - c^2| < \epsilon$.

Let $\delta = \frac{\epsilon}{|x+c|}$. Then we have

$$|x^2 - c^2| = |x - c||x + c| < \delta|x + c| = \frac{\epsilon}{|x + c|}|x + c| = \epsilon$$

so $f(x)$ is continuous at c.

(b) Is f uniformly continuous on $(0, 3)$? Fully justify your answer.

Solution: Yes, it is. Without doing the first part, we can observe that since x^2 is a polynomial, it is continuous everywhere. Further, since $[0, 3]$ is compact, every continuous function defined on $[0, 3]$ is also uniformly continuous on $[0, 3]$. Since $(0, 3) \subset [0, 3]$, f is uniformly continuous on $(0, 3)$.

Alternatively, observe that $\epsilon/(x + c)$ is at most $\epsilon/6$ for x and c between 0 and 3. Thus, we may take $\delta = \epsilon/6$ in the above argument to get a uniform bound on the entire interval.

$\boxed{\text{10 pts}}$ 6. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a nondecreasing function. Prove that at each point $c \in \mathbb{R}$, f is either continuous or has a jump discontinuity. (That is, show that no monotone function can have an essential or removable discontinuity at any point in its domain).

Solution: Fix some $c \in \mathbb{R}$, and let us assume (for now) that $\lim_{x\to c^-} f(x)$ exists. Then for any sequence $\{x_n\}$ with $x_n \to c$ and $x_n < c$, we know $f(x_n)$ converges to L. Since the limit exists, all such L must be the same number. Furthermore, since f is increasing, $L \leq f(c)$.

If $L = f(c)$, then f is continuous from below. If not, then f has a jump discontiuity at c.

Now let's see that $\lim_{x\to c^-} f(x)$ exists.

Let $L = \sup_{x<c}\{\, f(x) \,\}$. Since L is a least upper bound for this set, for any $\epsilon > 0$ there is an $x < c$ with $|f(x) - L| < \epsilon$. Since f is increasing, if $y > x$ we have $f(y) > f(x)$; this means that every point y in the interval (x, c) satisfies $|f(y) - L| < \epsilon$. That is, $\lim_{x\to c^-} f(x) = L$.

The argument above can be trivially modified to show that $\lim_{x\to c^+} f(x)$ exists and is no smaller than $f(x)$; call this limit M. If $L = M$, then f is continuous at C; otherwise, there is a jump discontinuity at c.

$\boxed{\text{10 pts}}$ 7. (a) Let f be continuous on $[a, b]$ with $f(x) \geq 0$ for all $x \in [a, b]$. Suppose that there exists $c \in (a, b)$ for which $f(c) > 0$. Prove that $\int_a^b f(x)\,dx > 0$.

Solution: Since f is continuous on $[a, b]$, it is integrable. Furthermore, since $f(c) > 0$ and f is continuous, there is an interval about c where $f(x)$ is positive, so the integral of f on that interval will also be positive.

More precisely, let $\epsilon = f(c)/2$. Since f is continuous, there is a δ so that for $x \in (c - \delta, c + \delta)$, $|f(x) - f(c)| < \epsilon$, that is, $f(x) > f(c)/2$. Thus,

$$\int_{c-\delta}^{c+\delta} f(x)\,dx > \int_{c-\delta}^{c+\delta} \frac{f(c)}{2}\,dx = 2\delta\frac{f(c)}{2} = \delta f(c) > 0.$$

Alternatively, you could assume that no such interval existed and see that you would have a contradiction to the Intermediate Value Theorem (f would have to jump from 0 to $f(c)$ without taking on values in between).

(b) Suppose f is nonnegative and integrable on $[a,b]$, and there exists $c \in (a,b)$ with $f(c) > 0$. Must it be true that $\int_a^b f(x)\,dx > 0$?
If so, give a proof; if not, give a counterexample.

Solution: No, it does not hold. Let $f(x) = \begin{cases} 0 & \text{if } x \neq \frac{1}{2} \\ 1 & \text{if } x = \frac{1}{2} \end{cases}$.
Then f is nonnegative and integrable on $[0,1]$ and $f(1/2) > 0$, but $\int_0^1 f(x)\,dx = 0$.

$\boxed{\text{10 pts}}$ 8. (a) Derive the Taylor series for $\ln(1+x)$. You may either derive it directly or via manipulation of another well-known series (e.g. the geometric series). For what x does the series converge?

Solution: Since $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, we have $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$. Integrating term by term gives

$$\ln(1+x) = c + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = c + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots,$$

but since $\ln(1+0) = \ln(1) = 0$, the constant c is 0.

This series converges for $x \in (-1, 1]$.
Since the original (geometric) series converges for $|x| < 1$ and diverges for $|x| > 1$, so does the new series. Further, when $x = 1$ we get the alternating harmonic series, which converges by the alternating series test. When $x = -1$, we have the harmonic series, which diverges.

Alternatively, you could derive the Taylor series directly. Taking derivatives gives you $f(0) = 0$, $f'(0) = 1$, $f''(0) = -1$, $f^{(3)}(0) = 2$, and in general, $f^{(n)}(0) = (-1)^{n+1}(n-1)!$. This gives the same series as above.
Then you can check the interval of convergence directly by using the ratio test, and check the two endpoints as above.

(b) Use the first two nonzero terms of the series to estimate $\ln(3/2)$.

Solution:

$$\ln(3/2) \approx \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{2^2} = \frac{3}{8}.$$

(c) Give an bound for the error in your answer to the previous part (and justify this bound).

Solution: From Taylor's remainder formula, we have that for $f(x) = \ln(1+x)$, the exact error in approximating $f(1/2)$ is

$$\frac{f^{(3)}(c)}{3!} \cdot \frac{1}{2^3} \text{ for some } c \in (0, 1/2).$$

Since $f^{(3)}(c) = 2/(1+c)^3$, the maximum occurs at $c = 0$, so the error is at most $\frac{1}{3} \cdot \frac{1}{8} = \frac{1}{24}$.
(In fact, $\ln(3/2) \approx 0.4055$ and $1/24 \approx 0.041667$ so this estimate is off by about 0.0305.)