MAT 513 Second Midterm

April 19, 2017

Name: ______________________ **ID:** __________________

Question:	1	2	3	4	5	6	Total
Points:	10	10	10	10	10	10	60
Score:							

There are 6 problems in this exam. Make sure that you have them all.

Do all of your work in this exam booklet, and cross out any work that the grader should ignore. You may use the backs of pages, but indicate what is where if you expect someone to look at it. **Books, calculators, extra papers, and discussions with friends are not permitted.** If the exam gets too stressful, maybe Kylie Jenner will show up with a Pepsi and make everything OK.

Points will be taken off for writing mathematically false statements, even if the rest of the problem is correct.

You have 90 minutes to complete this exam.

$\boxed{\text{5 points}}$ 1. (a) Suppose $f\colon A \to \mathbb{R}$. Define what it means for f to be **continuous on A**.
(There are several possible correct answers. Choose one.)

$\boxed{\text{5 points}}$ (b) Suppose $f\colon A \to \mathbb{R}$. Define what it means for f to be **uniformly continuous on A**.

$\boxed{\text{10 points}}$ 2. Compute the following limits using any correct method.

(a) $\lim_{x\to 0^+} x\ln(x)$

(b) $\lim_{x\to 1} \dfrac{x^2 - x^{-1}}{x-1}$

$\boxed{\text{10 points}}$ 3. Let $f(x) = \begin{cases} x^2 \sin(1/x^2) + x/2 & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$

(a) Compute $f'(0)$.

(b) Is there an interval $(-a, a)$ about zero on which $f(x)$ is increasing? Explain.

$\boxed{\text{10 points}}$ 4. Let $f_n(x) = (x - \frac{1}{n})^2$ for $x \in [0, 1]$. Does $\{f_n\}$ converge uniformly on $[0, 1]$? Fully justify your answer.

10 points

5. A function $g\colon A \to \mathbb{R}$ is an **open mapping** if for every open set $U \subseteq A$, its image $g(U)$ is open. Not all open mappings are continuous[1].

   Show that if $g\colon \mathbb{R} \to \mathbb{R}$ is an open mapping and and g is increasing, then it is continuous at every $x \in \mathbb{R}$.

[1] For example, let $\mathcal{C}$ be the middle-thirds Cantor set, and define $f(x) = 0$ when $x \in \mathcal{C} \cap (0,1)$. Let f send each of the open intervals $I_{a,n} = (a/3^n, (a+1)/3^n)$ in $(0,1) \setminus \mathcal{C}$ monotonically increasing onto the interval $(-1,1)$. Then f is an open mapping from $(0,1)$ to $(-1,1)$ but f is not continuous.

Extra Credit [5 pts]: on the back of this page, prove that f as described above is indeed an open mapping.

$\boxed{\text{10 points}}$ 6. Let g be a differentiable function defined on $[0, 2]$ with $g(0) = 1$, $g(1) = 1$ and $g(2) = 2$.

(a) Prove that at some point $c \in (0, 2)$, we have $g'(c) = 1/2$.

(b) Prove that at some point $b \in [0, 2]$, we have $g'(b) = 1/3$.