MAT 513 Solutions to Midterm 2

5 pts 1. (a) Suppose $f\colon A \to \mathbb{R}$. Define what it means for f to be **continuous on A**. (There are several possible correct answers. Choose one.)

Solution: The simplest is just the following:
For every $c \in A$, the limit $\lim_{x\to c} f(x)$ exists and is equal to $f(c)$.

However, you can replace the statement of limit above with any of the various equivalent definitions, for example:
For every $c \in A$ and for every $\epsilon > 0$, there exists a $\delta > 0$ so that for every $x \in A$ with $|x - c| < \delta$, we have $|f(x) - f(c)| < \epsilon$.

Or
A function f is continuous if for every open set $U \subseteq \mathbb{R}$, the set $f^{-1}(U)$ is also open.

5 pts (b) Suppose $f\colon A \to \mathbb{R}$. Define what it means for f to be **uniformly continuous on A**.

Solution: For every $\epsilon > 0$, there exists a $\delta > 0$ so that for every $c \in A$ and every $x \in A$ with $|x - c| < \delta$, we have $|f(x) - f(c)| < \epsilon$.

10 pts 2. Compute the following limits using any correct method.

(a) $\displaystyle\lim_{x\to 0^+} x\ln(x)$

Solution: Observe that $\lim_{x\to 0^+} \ln(x) = -\infty$ and $\lim_{x\to 0^+} x = 0$.
Rewriting and then using L'Hôpital's rule, we have

$$\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} x = 0$$

(b) $\displaystyle\lim_{x\to 1} \frac{x^2 - x^{-1}}{x-1}$

Solution: Here we can use algebra or L'Hôpital's rule. By algebra, we have

$$\frac{x^2 - x^{-1}}{x-1} = \frac{1}{x}\cdot\frac{x^3-1}{x-1} = \frac{x^2+x+1}{x} \qquad \text{provided } x \neq 1,$$

so

$$\lim_{x\to 1} \frac{x^2 - x^{-1}}{x-1} = \lim_{x\to 1} \frac{x^2+x+1}{x} = 3.$$

Alternatively, since the limit of the numerator and denominator are both 0, we can apply L'Hôpital's rule to get

$$\lim_{x\to 1} \frac{x^2 - x^{-1}}{x-1} = \lim_{x\to 1} \frac{2x + x^{-2}}{1} = 2 + 1 = 3.$$

$\boxed{\text{10 pts}}$ 3. Let $f(x) = \begin{cases} x^2 \sin(1/x^2) + x/2 & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$

(a) Compute $f'(0)$.

Solution: Using the definition of the derivative, we have

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x^2 \sin(1/x^2) + x/2}{x} = \lim_{x\to 0} x \sin(1/x^2) + 1/2 = 1/2,$$

using the fact that $-1 \le \sin(1/x^2) \le 1$ to see that $x \sin(1/x^2) \to 0$.

(b) Is there an interval $(-a, a)$ about zero on which $f(x)$ is increasing? Explain.

Solution: No.
Although $f'(0) > 0$, observe that for $x \neq 0$ we have

$$f'(x) = 2x \sin(1/x^2) - 2x^{-1} \cos(1/x^2) + 1/2.$$

In any interval about 0, f′(x) will take on all values in $\mathbb{R}$ (since as $x \to 0$, $\cos(1/x)$ changes sign infinitely often, and x^{-1} is unbounded). So $f(x)$ oscillates wildly as $x \to 0$, and cannot be said to be increasing on any interval containing 0.

$\boxed{\text{10 pts}}$ 4. Let $f_n(x) = (x - \frac{1}{n})^2$ for $x \in [0, 1]$. Does $\{f_n\}$ converge uniformly on $[0, 1]$? Fully justify your answer.

Solution: Clearly $\lim f_n(x) = x^2$ pointwise.

To see that the convergence is uniform, we need to show that for any $\epsilon > 0$, there is an N so that for all $n > N$, we have $|f_n(x) - x^2| < \epsilon$ for all $x \in [0, 1]$.

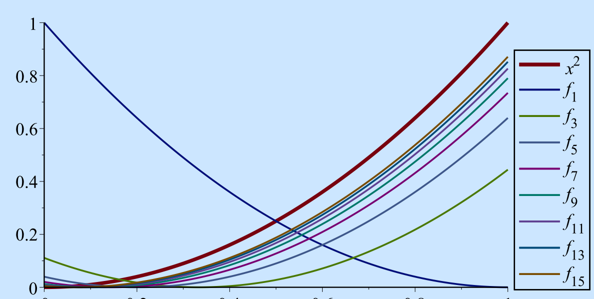
Let $\epsilon > 0$, and take $n > 2/\epsilon$. Then we have

$$\left| \left(x - \frac{1}{n} \right)^2 - x^2 \right| = \left| x^2 - \frac{2x}{n} + \frac{1}{n^2} - x^2 \right| = \frac{2x}{n} - \frac{1}{n^2} < \frac{2x}{n} < \frac{2x}{2/\epsilon} = x\epsilon \le \epsilon,$$

since $0 \le x \le 1$.

So the convergence is uniform on $[0, 1]$.

The uniform convergence should be apparent in the graph below.



10 pts 5. A function $g\colon A \to \mathbb{R}$ is an **open mapping** if for every open set $U \subseteq A$, its image $g(U)$ is open. Not all open mappings are continuous[1].

Show that if $g\colon \mathbb{R} \to \mathbb{R}$ is an open mapping and and g is increasing, then it is continuous at every $x \in \mathbb{R}$.

Solution: One way to do this is as follows:

Since g is increasing and open, it must be strictly increasing (if there were an interval (a, b) on which g were constant, then g would send the open set (a, b) to a point, which is not open). We've see that if g is a strictly monotone map with $g(\mathbb{R}) = B$, then it has an inverse map $g^{-1}\colon B \to \mathbb{R}$ (this inverse may not be continuous— but we will see that it is in this case).

We also saw that a map is continuous if and only if the preimage of open sets are open. Let U be any open set in $\mathbb{R}$. Then since g is open, $g(U)$ is also open. But since g is the inverse of g^{-1}, this means that for g^{-1}, preimages of open sets are open. So g^{-1} is continuous, and hence g is also continuous.

If you don't like that, here's another way:

Suppose g is discontinuous at some $c \in \mathbb{R}$. Then g cannot have a removable discontinuity at c; that is, if $\lim_{x\to c} g(x)$ exists, it must equal $g(c)$. This is because

$$\lim_{x\to c^-} g(x) \le g(c) \le \lim_{x\to c^+} g(x)$$

since g is increasing, yet the one-sided limits must be equal for the two-sided limit to exist.

Similarly, $g(x)$ cannot have an essential discontinuity at c, since for this to happen, one of the one-sided limits must not exist. To see this, let $L = \sup_{x<c}\{\, g(x) \,\}$. Since L is a least upper bound for this set, for any $\epsilon > 0$ there is an $x < c$ with $|g(x) - L| < \epsilon$. For every $y > x$ we have $g(y) > g(x)$; this means that every point y in the interval (x, c) satisfies $|g(y) - L| < \epsilon$. That is, $\lim_{x\to c^-} g(x) = L$. A similar argument shows that $\lim_{x\to c^+} g(x)$ must exist.

Suppose g has a jump discontinuity at c. Then either

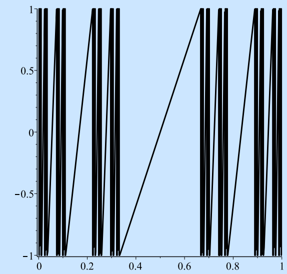
$$\lim_{x\to c^-} g(x) < g(c) \quad \text{or} \quad g(c) < \lim_{x\to c^+} g(x).$$

Now let $U = (c - \delta, c + \delta)$ for some small $\delta > 0$, and observe that $g(U)$ is not open, since $g(c)$ will not be interior to $g(U)$.

Since g cannot have a removable, jump, or essential discontinuity at c, it must be continuous at c. Since c was arbitrary, g must be continuous on all of $\mathbb{R}$.

[1] For example, let $\mathcal{C}$ be the middle-thirds Cantor set, and define $f(x) = 0$ when $x \in \mathcal{C} \cap (0, 1)$. Let f send each of the open intervals $I_{a,n} = (a/3^n, (a+1)/3^n)$ in $(0, 1) \smallsetminus \mathcal{C}$ monotonically increasing onto the interval $(-1, 1)$. Then f is an open mapping from $(0, 1)$ to $(-1, 1)$ but f is not continuous.

Extra Credit [5 pts]: on the back of this page, prove that f as described above is indeed an open mapping.



Extra Credit: The map f described in the footnote is clearly not continuous. For example, $\lim_{x\to\frac{2}{3}^-} f(x) = 1$, but $f(2/3) = 0$.
To see that f is an open mapping, let U be an open set.
If $U \cap \mathcal{C} = \emptyset$, then $f(U)$ will be an open subset of $(-1, 1)$. It might not be the whole interval, but since f is monotone on such pieces, $f(U)$ will be open.

Suppose instead U contains an interval (a, b) intersecting $\mathcal{C}$. Let x be a point in $\mathcal{C} \cap (a, b)$. Since every point $x \in \mathcal{C}$ is a limit point of $\mathcal{C}$, there is a sequence of points $\{x_n\}$ with $x_n \in \mathcal{C}$ that converges to x, and in particular we have another point $y \in \mathcal{C} \cap (a, b)$ with $y \neq x$.

Since $\mathcal{C}$ is totally disconnected, between any two points in $\mathcal{C}$ there will be an interval of the form $I_{a,n}$, so $f(U) = (-1, 1)$.

Thus, f is an open mapping, but is not continuous.

10 pts

6. Let g be a differentiable function defined on $[0, 2]$ with $g(0) = 1$, $g(1) = 1$ and $g(2) = 2$.

(a) Prove that at some point $c \in (0, 2)$, we have $g'(c) = 1/2$.

Solution: Since $g(0) = 1$ and $g(2) = 2$ and g is differentiable, we can apply the Mean Value Theorem to get the existence of a point $c \in (0, 2)$ for which

$$g'(c) = \frac{g(2) - g(0)}{2 - 0} = \frac{1}{2}.$$

(b) Prove that at some point $b \in [0, 2]$, we have $g'(b) = 1/3$.

Solution: Using the previous part, we found a point c with $g'(c) = 1/2$. Since $g(0) = g(1)$, we can also apply to Mean Value Theorem (or Rolle's Theorem, in this case) to find a point $a \in (0, 1)$ with $g'(a) = 0$. Although $g'(x)$ need not be continuous on $[0, 2]$, by Darboux's Theorem it cannot have a jump discontinuity. Thus, it takes on all values between 0 and $\frac{1}{2}$. In particular, there is a $b \in (a, c) \subseteq (0, 2)$ for which $g'(b) = 1/3$ (or, indeed, any number between 0 and $\frac{1}{2}$).