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Midterm Solutions

1. Let X and Y be topological spaces, and let $f: X \to Y$ be continuous and surjective. Show that if X is connected, then Y must be connected.

   If Y is not connected there exist two open, non empty sets A and B so that

   $A \cup B = Y$

   $A \cap B = \varnothing$

   - Since f is continuous $f^{-1}(A)$ and $f^{-1}(B)$ are open
   - Also $f^{-1}(A) \cap f^{-1}(B) = \varnothing$ since f is a function
     (i.e. if $x \in f^{-1}(A) \cap f^{-1}(B)$ then $f(x) \in A$, $f(x) \in B$, but $A \cap B = \varnothing$)
   - $f^{-1}(A) \cup f^{-1}(B) = X$ since f is surjective
     (i.e. every $y \in Y$ has some $x \in X$ so that $f(x) = y$ such a y is in A or B since $A \cup B = Y$)
   - Also $f^{-1}(A) \neq \varnothing$ since $A \neq \varnothing$ and $f^{-1}(B) \neq \varnothing$ since $B \neq \varnothing$.

   Therefore X is not connected.

2. Let M be the image of R^2 under $h: R^2 \to R^3$ given by $h(x, y) = (x^3, x^2, y)$

a) Calculate Dh.

$$Dh = \begin{pmatrix} 3x^2 & 0 \\ 2x & 0 \\ 0 & 1 \end{pmatrix}$$

b) What is the tangent space to M at (0, 0, 0)?

$$Dh_{(0,0,0)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = R$$

c) What is the tangent space to M at (1, 1, 1)?

$$Dh_{(1,1,1)} = \begin{pmatrix} 3 & 0 \\ 2 & 0 \\ 0 & 1 \end{pmatrix} = R^2$$

d) Is M a smooth manifold? Justify your answer.
No, M is not a smooth manifold because the derivative matrix is not of full rank at (0,0,0).

3. The set of all 2×2 matricies with real entries and determinant 1 is called $SL_2(R)$. Show that $SL_2(R)$ is a smooth 3-manifold by giving charts for it. Be sure to justify that your charts cover $SL_2(R)$ and that they are smooth.

[Note: We can show that $SL_2(R)$ is a manifold since det is a smooth function and $SL_2(R)$ is the preimage under det.]

$M_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in R \right\}$ is diffeomorphic to R^4.

$\det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = 1$ then $ad - bc = 1$

Solve for b then $b = \dfrac{ad-1}{c}$ if $c \neq 0$

The chart for $c \neq 0$ from $R^3 \to SL_2(R)$ is

$(a,c,d) \to \begin{pmatrix} a & \dfrac{ad-1}{c} \\ c & d \end{pmatrix}$ This chart is continuous and smooth.

The chart for $b \neq 0$

$(a,b,d) \to \begin{pmatrix} a & b \\ \dfrac{ad-1}{b} & d \end{pmatrix}$ This chart is continuous and smooth.

If $b = 0$ and $c = 0$

$(a,0,d) \to \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ with $d = \dfrac{1}{a}$ and $a \neq 0$. This chart is continuous and smooth.

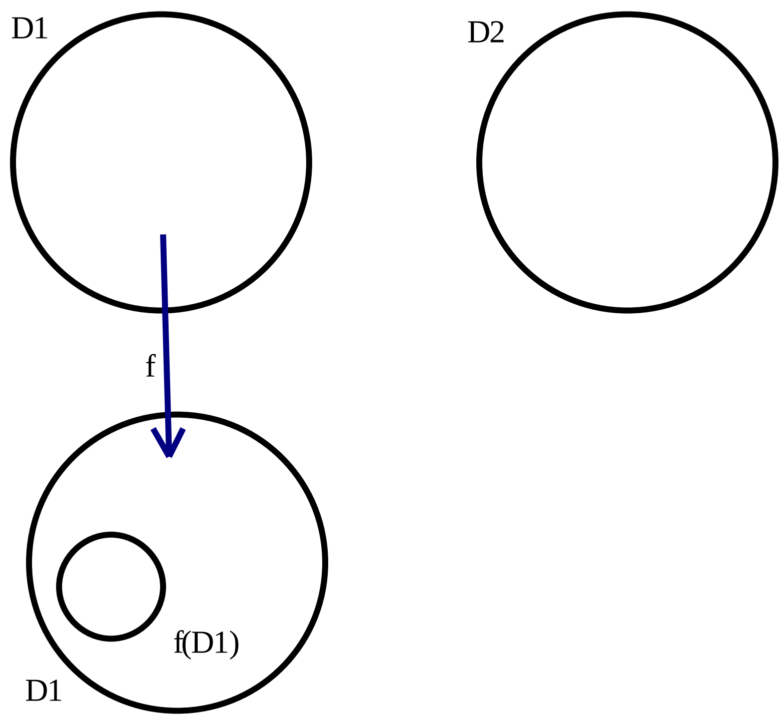
4. Consider the two disjoint, closed disks

$$D_1 = \{(x, y) \mid x^2 + y^2 \le 1\} \text{ and } D_2 = \{(x, y) \mid (x-4)^2 + y^2 \le 1\}$$

Let $M = D_1 \cup D_2$.
Suppose $f : M \to M$ a smooth function. Show that f must have a period 2 point; that is, there must be a point $p \in M$ such that $f(f(p)) = p$.

Case I:

a) $f(D_1) \subset D_1$



For this case we ignore D_2.

Apply Brouer's Fixed Point Theorem, then there exists p such that $f(p) = p$ then $f(f(p)) = f(p) = p$

b) $f(D_2) \subset D_2$, then $p \in D_2$.

Case II: If neither Case I a) nor I b) hold, since f is continuous the following must hold

- $f(D_1) \subset D_2$ then $f(f(D_1)) \subset f(D_2) \subset D_1$ so $f^2 : D_1 \to D_1$
  There exists $p \in D_1$ with $f(f(p)) = p$
- $f(D_2) \subset D_1$ then $f(f(D_2)) \subset f(D_1) \subset D_2$ so $f^2 : D_2 \to D_2$
  There exists $p \in D_2$ with $f(f(p)) = p$