MAT 364 Topology Notes

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We want to prove that, regardless of the size of the sphere, index at an isolatex zero stays the same as long as the sphere contains only that one zero. More generally, and this is what the book has, the "concept of index is invariant under diffeomorphism of U." We're at §6 of the Milnor book.

We're looking at the index of a vector field at an isolated zero. That is, on our n-manifold M, given a vector field $V : M \to TM$, we're studying what happens some z where $V(z) = 0$ and for a small enough neighborhood $D_\varepsilon(z)$, only z has a zero vector. (That is, z is an isolated zero.)

Using any of those D_ε neighborhoods described above, we may define the **index** of V at z (denoted $\imath(V; z)$) as the degree from $\partial D_\varepsilon(z) \to S^{n-1}$ of the normalized version of our vector field V. Since M is an n-manifold, $\partial D_\varepsilon \cong S^{n-1}$. Thus, with a 2-manifold, the vector field maps a circle to a circle and we may think of the degree as that on $\mathbb{T}^2$.

¡PIX: Mapping from circle to circle.¿

Recall: two maps are homotopic if there are no zeros between the spheres. Suppose $f : M \to N$ is a smooth diffeomorphism, with v_M being a vector field on M, v_N a vector field on N.

Definition. *Two vector fields v_M and v_N are **corresponding vector fields under** f when df_x takes $v_M(x)$ to $v_n(f(x))$ for each $x \in M$. Thus, $v_N = df \circ v_m \circ f^{-1}$.*

Let's focus on some particular zero, z. Without loss of generality, $z = 0$ in any set U. We want to prove the following lemma: If V is a vector field, and some open U has an isolated zero at z, and there's a corresponding vector field V' with and isloated zero z, then $\imath(V', z) = \imath(V, f^{-1}(z))$. We'll first prove the more general lemma: any orientation-preserving diffeomorphism f of $\mathbb{R}^n$ is smoothly isotopic to the identity.

Proof: WLOG, assume $f(0) = 0$. Since f is smooth, $df_0 : TM_0 \to TM_0$ exists, and $df_0 = \lim_{t\to 0} \frac{f(0+tx)-f(0)}{t} = \lim_{t\to 0} \frac{f(tx)}{t}$. Recall that, since f is a diffeomorphism of $\mathbb{R}^n, TM_0 \cong \mathbb{R}^n$.

Define an isotopy $F : \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n$ by $F(x, t) = \frac{f(xt)}{t}$. Note that $F(x, 1) = f(x)$ and $F(x, 0) = df_0(x)$. So, f is homotopic to its derivative. However, since f is a diffeomorphism, the df is homotopic to the identity because df is isomorphic anywhere. Therefore, f is isotopic to the identity. The proof that it's smoothly isotopic is in Milnor, page 34. Basically, we can break it down into a bunch of smooth functions along the coordinates.

Now, we can prove the other lemma.

Homotopic vector fields from one sphere to another. We have two corresponding vector fields v, v' on two different open sets U, U'. There is a smooth $f : U \to U'$.

Case 1: f is orientation preserving. Define $f_t : U \to \mathbb{R}^n$ so that $f_0 = \mathrm{id}$ and $f_1 = f$. We can do this because any orientation preserving diffeomorphis m is smoothly isotopic to the identity. This means we can also define $v_t = df_t \circ v \circ f_t^{-1}$. Then because $f_0 = \mathrm{id}, v_0 = v$ and because $f_1 = f, v_1 = v'$. Moreover, those vector fields on v_t are well defined and nonzero everywhere on the sphere **BECAUSE z IS AN ISOLATED ZERO!** Hence, the index of v at 0 must be equal to the index at v' at 0.

Case 2: f is oreintation reserving. Let R be a reflection so that $R \circ f$ is orientation preserving. Apply case 1. R doesn't change the degree, so $\deg(R \circ f) = \deg(f)$. Thus we have proven that $\imath$ does not depend on the size of the sphere, just the vector field and isolated zero.

Now we switch to Euler Characteristic. We need the sets of vertices and edges to define a graph, which in turn will define the Euler Characteristic. On some smooth 2-manifold M, the set of vertices is some $V \subseteq M$ such that, for each $v \in V$, there exists a neighborhood W so that $V \cap W = \{v\}$ (i.e. each vertex is isolated). Let $\gamma_a^b : [0, 1] \to M$ be a smooth, continuous function such that $\gamma_a^b(0) = a$ and $\gamma_a^b(1) = b$. We call the set of paths $P = \{\gamma_a^b : \quad a, b \in M\}$. Then our set of edges may be any $E \subseteq P$ such that the paths only intersect at the vertices (i.e. $\cap E \subseteq V$).

A **graph embedded on** M is any two sets V, E that satisfy the above qualities. A **triangulation of** M is any connected graph embedded on M such that there exists a diffeomorphism between a finite union of disjoint open sets in $\mathbb{R}^2$ and $M - (V \cup E)$. That is, if (V, E) is a triangulation, there exists some $F = \{U : \quad U \subseteq \mathbb{R}^2, U \text{ is open}\}, |F| = f, \cap F =$ for some $f \in \mathbb{N}$ and there also exists some diffeomorphism $g : F \to M - (V \cup E)$.

The open sets in F chart to faces. We define $\chi(M) = |V| - |E| + |F|$ for any triangulation. We would like to prove that the triangulation does not matter.